

THE SHARP REMEZ-TYPE INEQUALITY FOR EVEN TRIGONOMETRIC POLYNOMIALS ON THE PERIOD

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ABSTRACT. We prove that

$$\max_{t \in [-\pi, \pi]} |Q(t)| \leq T_{2n}(\sec(s/4)) = \frac{1}{2}((\sec(s/4) + \tan(s/4))^{2n} + (\sec(s/4) - \tan(s/4))^{2n})$$

for every even trigonometric polynomial Q of degree at most n with complex coefficients satisfying

$$m(\{t \in [-\pi, \pi] : |Q(t)| \leq 1\}) \geq 2\pi - s, \quad s \in (0, 2\pi),$$

where $m(A)$ denotes the Lebesgue measure of a measurable set $A \subset \mathbb{R}$ and T_{2n} is the Chebysev polynomial of degree $2n$ on $[-1, 1]$ defined by $T_{2n}(\cos t) = \cos(2nt)$ for $t \in \mathbb{R}$. This inequality is sharp. We also prove that

$$\max_{t \in [-\pi, \pi]} |Q(t)| \leq T_{2n}(\sec(s/2)) = \frac{1}{2}((\sec(s/2) + \tan(s/2))^{2n} + (\sec(s/2) - \tan(s/2))^{2n})$$

for every trigonometric polynomial Q of degree at most n with complex coefficients satisfying

$$m(\{t \in [-\pi, \pi] : |Q(t)| \leq 1\}) \geq 2\pi - s, \quad s \in (0, \pi).$$

0. FOREWORD

I started my Ph.D. program in June, 1987, at The Ohio State University, as a student of Paul Nevai. In August, 1987, Paul Nevai moved from Columbus, Ohio, to Columbia, South Carolina, to spend the school year 1987–88 at the University of South Carolina, and each of his students followed him. This is how I met Mr. (Yingkang) Hu, who was a student of Ron DeVore at that time. It was a somewhat short but very exciting time for me at the University of South Carolina. Not only had I passed each of my necessary exams in my Ph.D. program but I had the possibility to learn from and interact with some of the very best researchers in approximation theory. My office neighbors were Paul Nevai, Ron DeVore, and George Lorentz. George Lorentz happened to be Ron DeVore's

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office neighbor on one hand and my office neighbor on the other hand to finalize the book entitled “Constructive Approximation” he was writing jointly with Ron DeVore. It was a privilege for me that the authors of this fundamental book in approximation theory asked me to read some sections of their book in preparation and took my remarks seriously. I was also lucky to have talented, motivated, and helpful class-mates such as Yingkang Hu, George Kyriazis, Jiaxiang (John) Zhang, and Shing-Whu Jha around. During the school year 1987–88 Yingkang was a good friend of mine. We spent much time together as class-mates as well as friends. I liked to eat Chinese food in the company of Mr. Hu, Jiaxiang Zhang, Shing-Whu Jha, and some of my other Chinese friends. We all took Paul Nevai’s class on Orthogonal Polynomials. We traveled quite much together in this school year to attend various conferences, but Yinkang, Jiaxiang, and me traveled together as tourists as well in the spring break. In August, 1988, Paul Nevai moved back to Columbus, and I followed him to write my dissertation at The Ohio State University. I defended my Ph.D. dissertation at the University of South Carolina in May, 1989. A central part of my dissertation was a Remez-type inequality for (generalized) algebraic and trigonometric polynomials and their applications in the proofs of various other inequalities for generalized polynomials. I was not able to prove the sharp Remez-type inequality for trigonometric polynomials, but I have managed to prove the sharp upper bound for $|Q(0)|$ for even trigonometric polynomials Q satisfying

$$m\{t \in (0, 2\pi] : |Q(t)| \leq 1\} \geq 2\pi - s, \quad s \in (0, 2\pi),$$

where $m(A)$ denotes the Lebesgue measure of a set $A \subset \mathbb{R}$. After some 30 years I have revisited the topic of my dissertation. While the sharp Remez-type inequality for trigonometric polynomials remains open, I am glad to prove the sharp Remez-type inequality at least for even trigonometric polynomials in this volume dedicated to the memory of Dr. Hu. After defending my thesis I met Yingkang very rarely, but we were keeping in touch. In March, 2002, Yingkang visited us at Texas A&M University, and gave a talk in our Center of Approximation Theory Seminar with the title “Equivalence of Moduli of Smoothness”, a topic on which he published two papers in Journal of Approximation Theory. During his visit we watched the Academy Award Ceremony in my apartment as he was very much interested in some of the latest movies. This may have been the last time I saw him and I was very sad to learn about his death in March, 2016.

1. INTRODUCTION

Let \mathcal{T}_n denote the set of all real trigonometric polynomials of degree at most n . Let \mathcal{T}_n^c denote the set of all complex trigonometric polynomials of degree at most n . Let $K := \mathbb{R} \pmod{2\pi}$. Let $m(A)$ denote the Lebesgue measure of a measurable set $A \subset \mathbb{R}$. For $s \in (0, 2\pi)$ let

$$\mathcal{T}_n(s) := \{Q \in \mathcal{T}_n : m(\{t \in K : |Q(t)| \leq 1\}) \geq 2\pi - s\}$$

and

$$\mathcal{T}_n^c(s) := \{Q \in \mathcal{T}_n^c : m(\{t \in K : |Q(t)| \leq 1\}) \geq 2\pi - s\}.$$

Let \mathcal{P}_n denote the set of all algebraic polynomials of degree at most n with real coefficients. Let \mathcal{P}_n^c denote the set of all algebraic polynomials of degree at most n with complex coefficients. Let T_n be the Chebyshev polynomial of degree n on $[-1, 1]$ defined by $T_n(\cos t) = \cos(nt)$ for $t \in K$. For real numbers $a < b$ and a Lebesgue measurable set $A \subset [a, b]$ let

$$\mu_{[a,b]}(A) := \int_A \frac{(b-a)/2}{\sqrt{((b-a)/2)^2 - (x - (a+b)/2)^2}} dx.$$

The classical Remez inequality states that if $P \in \mathcal{P}_n$, $s \in (0, 2)$, and

$$m(\{x \in [-1, 1] : |P(x)| \leq 1\}) \geq 2 - s,$$

then

$$\max_{x \in [-1, 1]} |P(x)| \leq T_n \left(\frac{2+s}{2-s} \right),$$

where T_n defined by

$$T_n(x) := \cos(n \arccos x), \quad x \in [-1, 1],$$

is the Chebyshev polynomial of degree n . This inequality is sharp and

$$T_n \left(\frac{2+s}{2-s} \right) \leq \exp(\min\{5ns^{1/2}, 2n^2s\}), \quad s \in (0, 1].$$

Remez-type inequalities for various classes of functions have been studied by several authors, and they have turned out to be applicable and connected to various problems in approximation theory. See [1]–[34], for example.

In [16] we proved that

$$(1.1) \quad |Q(0)| \leq T_{2n}(\sec(s/4))$$

for every even $Q \in \mathcal{T}_n^c(s)$ and $s \in (0, 2\pi)$. However, 0 is a special point in the study of even trigonometric polynomials $Q \in \mathcal{T}_n^c(s)$, so the question whether or not

$$(1.2) \quad \max_{t \in K} |Q(t)| \leq T_{2n}(\sec(s/4))$$

holds at least for all even trigonometric polynomials $Q \in \mathcal{T}_n^c(s)$ remained open, while it was speculated that (1.2) may hold for all $Q \in \mathcal{T}_n^c(s)$. In this paper we show that (1.2) holds for all even $Q \in \mathcal{T}_n^c(s)$, while it remains open whether or not (1.2) holds for all $Q \in \mathcal{T}_n^c(s)$.

2. NEW RESULTS

Our first result is a sharp Remez-type inequality for even trigonometric polynomials with complex coefficients.

Theorem 2.1. *We have*

$$\max_{t \in K} |Q(t)| \leq T_{2n}(\sec(s/4)) = \frac{1}{2}((\sec(s/4) + \tan(s/4))^{2n} + (\sec(s/4) - \tan(s/4))^{2n})$$

for every even $Q \in \mathcal{T}_n^c(s)$ and $s \in (0, 2\pi)$. Equality holds if and only if

$$\{t \in K : |Q(t)| \leq 1\} = [s/2, 2\pi - s/2]$$

and $Q \in \mathcal{T}_n(s)$ is of the form

$$Q(t) = T_{2n} \left(\frac{\cos(t/2)}{\cos(s/4)} \right).$$

Our next result is a Remez-type inequality for odd trigonometric polynomials with complex coefficients.

Theorem 2.2. *We have*

$$\max_{t \in K} |Q(t)| \leq \frac{1}{2}((\sec(s/4) + \tan(s/4))^{2n} + (\sec(s/4) - \tan(s/4))^{2n}) + \frac{1}{\sqrt{2}}$$

for every odd $Q \in \mathcal{T}_n^c(s)$ and $s \in (0, 2\pi)$.

Theorem 2.1 implies the following result for all trigonometric polynomials with complex coefficients.

Theorem 2.3. *We have*

$$\max_{t \in K} |R(t)| \leq T_{2n}(\sec(s/2)) = \frac{1}{2}((\sec(s/2) + \tan(s/2))^{2n} + (\sec(s/2) - \tan(s/2))^{2n})$$

for every $R \in \mathcal{T}_n^c(s)$ and $s \in (0, \pi)$.

Note that Theorem 2.2 was proved in [16] only for all $R \in \mathcal{T}_n(s)$ (rather than $R \in \mathcal{T}_n^c(s)$) and $s \in (0, \pi)$. It remains open whether or not Theorem 2.1 can be extended to all $Q \in \mathcal{T}_n^c(s)$ and $s \in (0, 2\pi)$.

3. LEMMAS

Our first lemma is the conclusion (13) of Section 7 in [16]. This deep result plays a central role in the proof of Theorem 2.1.

Lemma 3.1. *We have*

$$|U(0)| \leq T_{2n}(\sec(s/4))$$

for every even $U \in \mathcal{T}_n(s)$ and $s \in (0, 2\pi)$. Equality holds if and only if

$$\{t \in K : |U(t)| \leq 1\} = [s/2, 2\pi - s/2]$$

and

$$U(t) = T_{2n} \left(\frac{\cos(t/2)}{\cos(s/4)} \right).$$

Our next lemma extends Lemma 3.1 to all even trigonometric polynomials with complex coefficients.

Lemma 3.2. *We have*

$$|Q(0)| \leq T_{2n}(\sec(s/4))$$

for every even $Q \in \mathcal{T}_n^c(s)$ and $s \in (0, 2\pi)$. Equality holds if and only if

$$\{t \in K : |Q(t)| \leq 1\} = [s/2, 2\pi - s/2]$$

and

$$Q(t) = T_{2n} \left(\frac{\cos(t/2)}{\cos(s/4)} \right).$$

Proof. Let $Q \in \mathcal{T}_n^c(s)$. Choose $c \in \mathbb{C}$ with $|c| = 1$ so that $cQ(0)$ is real. Define $U \in \mathcal{T}_n(s)$ by

$$U(t) := \operatorname{Re}(cQ(t)), \quad t \in K.$$

Applying Lemma 3.1 to $U \in \mathcal{T}_n(s)$ we obtain

$$|Q(0)| = |S(0)| \leq T_n(\sec(s/4)),$$

and equality holds if and only if

$$\{t \in K : |Q(t)| \leq 1\} = [-\pi + s/2, \pi - s/2]$$

and

$$Q(t) = T_{2n} \left(\frac{\cos(t/2)}{\cos(s/4)} \right).$$

□

Observe that every even $U \in \mathcal{T}_n^c$ can be written as $U(t) = P(\cos t)$, where $P \in \mathcal{P}_n^c$. In terms of algebraic polynomials Lemma 3.1 can be formulated as follows.

Lemma 3.3. *We have*

$$|P(1)| \leq T_{2n}(\sec(s/4))$$

for every $P \in \mathcal{P}_n^c$ such that

$$\mu_{[-1,1]}(\{x \in [-1, 1] : |P(x)| \leq 1\}) \geq \pi - s/2.$$

Equality holds if and only if

$$\{x \in [-1, 1] : |P(x)| \leq 1\} = [-1, \cos(s/2)]$$

and

$$P(\cos t) = T_{2n} \left(\frac{\cos(t/2)}{\cos(s/4)} \right).$$

Associated with $P \in \mathcal{P}_n^c$ let $\tilde{P} \in \mathcal{P}_n^c$ be defined by $\tilde{P}(x) = P(-x)$. Observing that

$$\mu_{[-1,1]}(\{x \in [-1, 1] : |P(x)| \leq 1\}) = \mu_{[-1,1]}(\{x \in [-1, 1] : |\tilde{P}(x)| \leq 1\})$$

we obtain the following.

Lemma 3.3*. *We have*

$$|P(-1)| \leq T_{2n}(\sec(s/4))$$

for every $P \in \mathcal{P}_n^c$ such that

$$\mu_{[-1,1]}(\{x \in [-1, 1] : |P(x)| \leq 1\}) \geq \pi - s/2.$$

Equality holds if and only if

$$\{x \in [-1, 1] : |P(x)| \leq 1\} = [-\cos(s/2), 1]$$

and

$$P(-\cos t) = T_{2n}\left(\frac{\cos(t/2)}{\cos(s/4)}\right).$$

We need to transform the above two lemmas linearly from the interval $[-1, 1]$ to the interval $[a, b]$.

Lemma 3.4. *We have*

$$|P(b)| \leq T_{2n}(\sec(s/4))$$

for every $P \in \mathcal{P}_n^c$ such that

$$\mu_{[a,b]}(\{x \in [a, b] : |P(x)| \leq 1\}) \geq \frac{b-a}{2} (\pi - s/2).$$

Equality holds if and only if

$$\{x \in [a, b] : |P(x)| \leq 1\} = [a, (a+b)/2 + ((b-a)/2) \cos(s/2)]$$

and

$$P(((b-a)/2) \cos t + (a+b)/2) = T_{2n}\left(\frac{\cos(t/2)}{\cos(s/4)}\right).$$

Lemma 3.4*. *We have*

$$|P(a)| \leq T_{2n}(\sec(s/4))$$

for every $P \in \mathcal{P}_n^c$ such that

$$\mu_{[a,b]}(\{x \in [a, b] : |P(x)| \leq 1\}) \geq \frac{b-a}{2} (\pi - s/2).$$

Equality holds if and only if

$$\{x \in [a, b] : |P(x)| \leq 1\} = [(a+b)/2 - ((b-a)/2) \cos(s/2), b]$$

and

$$P((- (b-a)/2) \cos t + (a+b)/2) = T_{2n}\left(\frac{\cos(t/2)}{\cos(s/4)}\right).$$

Observing that $P \in \mathcal{P}_n^c$ implies $\tilde{P} \in \mathcal{P}_n^c$, where $\tilde{P}(x) = P(-x)$ we obtain the following.

Lemma 3.5. *We have*

$$\mu_{[1-2r,1]}(A) > r^{1/2}\mu_{[-1,1]}(A)$$

for every $r \in (0, 1)$ and for every Lebesgue measurable set $A \subset [1 - 2r, 1]$ with $m(A) > 0$.

Proof. Let $r \in (0, 1)$ and let $A \subset [1 - 2r, 1]$ be a Lebesgue measurable set with $m(A) > 0$. We have

$$\begin{aligned} \frac{\mu_{[1-2r,1]}(A)}{\mu_{[-1,1]}(A)} &= \frac{\int_A \frac{r}{\sqrt{r^2 - (x - (1-r))^2}} dx}{\int_A \frac{1}{\sqrt{1-x^2}} dx} = \frac{\int_A \frac{r}{\sqrt{(1-x^2)}} \frac{\sqrt{(1-x^2)}}{\sqrt{r^2 - (x - (1-r))^2}} dx}{\int_A \frac{1}{\sqrt{1-x^2}} dx} \\ &> \frac{\int_A \frac{r}{\sqrt{1-x^2}} dx \min_{y \in [1-2r,1]} \frac{\sqrt{1-y^2}}{\sqrt{r^2 - (y - (1-r))^2}}}{\int_A \frac{1}{\sqrt{1-x^2}} dx} \\ &\geq rr^{-1/2} = r^{1/2} \end{aligned}$$

as

$$\min_{y \in [1-2r,1]} \frac{\sqrt{1-y^2}}{\sqrt{r^2 - (y - (1-r))^2}} = \min_{y \in [1-2r,1]} \frac{\sqrt{1+y}}{\sqrt{y - (1-2r)}} = \frac{\sqrt{2}}{\sqrt{2r}} = r^{-1/2}.$$

□

Lemma 3.6. *We have*

$$\mu_{[-1,1-2r]}(A) > (1-r)^{1/2}\mu_{[-1,1]}(A)$$

for every $r \in (0, 1)$ and for every Lebesgue measurable set $A \subset [-1, -1+2r]$ with $m(A) > 0$.

Proof. Let $r \in (0, 1)$ and let $A \subset [-1, -1+2r]$ be a Lebesgue measurable set with $m(A) > 0$. Applying Lemma 3.5 with A replaced by $-A := \{-x : x \in A\} \subset [1 - 2(1-r), 1]$ and $r \in (0, 1)$ replaced by $1-r \in (0, 1)$, we get

$$\mu_{[-1,-1+2r]}(A) = \mu_{[1-2(1-r),1]}(-A) > (1-r)^{1/2}\mu_{[-1,1]}(-A) = (1-r)^{1/2}\mu_{[-1,1]}(A).$$

□

Lemma 3.7. *We have*

$$\arccos(1-2r) > \pi r^{1/2} \quad \text{and} \quad \pi - \arccos(1-2r) > \pi(1-r)^{1/2}$$

for all $r \in (0, 1)$.

Proof. The first inequality is equivalent to $\sin t > (2/\pi)t$ for all $t = (\pi/2)r^{1/2} \in (0, \pi/2)$. The second inequality follows from the first one as

$$\pi - \arccos(1-2r) = \arccos(-1+2r) = \arccos(1-2(1-r)) > \pi(1-r)^{1/2}$$

for all $r \in (0, 1)$. □

4. PROOF OF THEOREM 2.1

Proof of Theorem 2.1. Let $Q \in \mathcal{T}_n^c(s)$ be even with $s \in (0, 2\pi)$. Then Q is of the form $Q(t) = P(\cos t)$, where $P \in \mathcal{P}_n^c$ satisfies

$$(4.1) \quad \mu_{[-1,1]}(\{x \in [-1, 1] : |P(x)| \leq 1\}) \geq \pi - s/2.$$

We want to prove that

$$(4.2) \quad |P(\alpha)| \leq T_{2n}(\sec(s/4))$$

for all $\alpha \in [-1, 1]$.

If $\alpha = 1$ then (4.2) holds by Lemma 3.3. If $\alpha = -1$ then (4.2) holds by Lemma 3.3*. Assume now that $\alpha = 1 - 2r \in (-1, 1)$, that is, $\alpha = 1 - 2r$ with $r \in (0, 1)$. It follows from (4.1) and the definition of the measure $\mu_{[-1,1]}$ that

$$m(\{t \in [0, \pi] : |P(\cos t)| \leq 1\}) \geq \pi - s/2,$$

and hence we have either

$$(4.3) \quad m(\{t \in [0, \arccos(1 - 2r)] : |P(\cos t)| \leq 1\}) \geq \frac{\pi - \arccos(1 - 2r)}{\pi}(\pi - s/2)$$

or

$$(4.4) \quad m(\{t \in [\arccos(1 - 2r), \pi] : |P(\cos t)| \leq 1\}) \geq \frac{\arccos(1 - 2r)}{\pi}(\pi - s/2).$$

For the sake of brevity let

$$(4.5) \quad A_1 := \{x \in [-1, 1 - 2r] : |P(x)| \leq 1\} \quad \text{and} \quad A_2 := \{x \in [1 - 2r, 1] : |P(x)| \leq 1\}.$$

By (4.3), (4.4), and (4.5) we have either

$$(4.6) \quad \mu_{[-1,1]}(A_1) \geq \frac{\pi - \arccos(1 - 2r)}{\pi}(\pi - s/2)$$

or

$$(4.7) \quad \mu_{[-1,1]}(A_2) \geq \frac{\arccos(1 - 2r)}{\pi}(\pi - s/2)$$

Suppose that (4.6) holds. Then, combining Lemmas 3.6 and 3.7 we obtain

$$\begin{aligned} \mu_{[-1,1-2r]}(A_1) &> (1 - r)^{1/2} \mu_{[-1,1]}(A_1) \geq (1 - r)^{1/2} \frac{\pi - \arccos(1 - 2r)}{\pi} (\pi - s/2) \\ &> (1 - r)^{1/2} (1 - r)^{1/2} (\pi - s/2) \geq (1 - r)(\pi - s/2), \end{aligned}$$

and

$$|P(\alpha)| = |P(1 - 2r)| < T_{2n}(\sec(s/4))$$

follows from Lemma 3.4.

Suppose that (4.7) holds. Then, combining Lemmas 3.5 and 3.7 we obtain

$$\begin{aligned} \mu_{[1-2r,1]}(A_2) &> r^{1/2} \mu_{[-1,1]}(A_2) \geq r^{1/2} \frac{\arccos(1-2r)}{\pi} (\pi - s/2) \\ &> r^{1/2} r^{1/2} (\pi - s/2) \geq r(\pi - s/2), \end{aligned}$$

and

$$|P(\alpha)| = |P(1 - 2r)| < T_{2n}(\sec(s/4))$$

follows from Lemma 3.4*. \square

Proof of Theorem 2.2. Let $Q \in \mathcal{T}_n^c(s)$ be odd. Then R defined by $R(t) = 2|Q(t)|^2 - 1$ is even and $R \in \mathcal{T}_{2n}^c(s)$. Applying Theorem 2.1 to R we obtain

$$\max_{t \in K} |2|Q(t)|^2 - 1| = \max_{t \in K} |R(t)| \leq \frac{1}{2} ((\sec(s/4) + \tan(s/4))^{4n} + (\sec(s/4) - \tan(s/4))^{4n}),$$

and hence

$$\begin{aligned} \max_{t \in K} |Q(t)| &\leq \left(\frac{1}{4} ((\sec(s/4) + \tan(s/4))^{4n} + (\sec(s/4) - \tan(s/4))^{4n}) + \frac{1}{2} \right)^{1/2} \\ &\leq \frac{1}{2} ((\sec(s/4) + \tan(s/4))^{2n} + (\sec(s/4) - \tan(s/4))^{2n}) + \frac{1}{\sqrt{2}} \end{aligned}$$

\square

Proof of Theorem 2.3. Observe that if $R \in \mathcal{T}_n^c(s)$, $s \in (0, \pi)$, and Q is defined by $Q(t) = \frac{1}{2}(R(t) + R(-t))$, then $R \in \mathcal{T}_n^c(2s)$. Hence Theorem 2.1 implies that

$$(4.8) \quad |R(0)| = |Q(0)| \leq T_{2n}(\sec(s/2))$$

holds for every even $R \in \mathcal{T}_n^c(s)$ and $s \in (0, \pi)$. The theorem now follows by a simple shift. Namely, observe that $R \in \mathcal{T}_n^c(s)$ implies that $R_a \in \mathcal{T}_n^c(s)$ where R_a is defined by $R_a(t) := R(t+a)$ for $t \in K$ and $a \in K$. Hence (4.8) implies that

$$|R(a)| = |R_a(0)| \leq T_{2n}(\sec(s/2))$$

holds for every even $R \in \mathcal{T}_n^c(s)$, $s \in (0, \pi)$, and $a \in K$. \square

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