POINTWISE REMEZ- AND NIKOLSKII-TYPE INEQUALITIES FOR EXPONENTIAL SUMS

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Abstract. Let

$$E_n := \left\{ f : f(t) = a_0 + \sum_{j=1}^{n} a_j e^{\lambda_j t} \,, \quad a_j, \lambda_j \in \mathbb{R} \right\}.$$

So E_n is the collection of all $n+1$ term exponential sums with constant first term. We prove the following two theorems.

Theorem 1 (Remez-Type Inequality for E_n at 0). *Let $s \in \left(0, \frac{1}{2}\right]$. There are absolute constants $c_1 > 0$ and $c_2 > 0$ such that*

$$\exp(c_1 ns) \leq \sup_f |f(0)| \leq \exp(c_2 ns)\,,$$

where the supremum is taken for all $f \in E_n$ satisfying

$$m\left(\{x \in [-1,1] : |f(x)| \leq 1\}\right) \geq 2 - s\,.$$

Theorem 2 (Nikolskii-Type Inequality for E_n). *There are absolute constants $c_1 > 0$ and $c_2 > 0$ such that*

$$c_1^{1+1/q} \left(\frac{1+qn}{\min\{y-a, b-y\}} \right)^{1/q} \leq \sup_{0 \neq f \in E_n} \frac{|f(y)|}{\|f\|_{L_q[a,b]}} \leq \left(\frac{c_2(1+qn)}{\min\{y-a, b-y\}} \right)^{1/q}$$

for every $a < y < b$ and $q > 0$.

It is quite remarkable that, in the above Remez- and Nikolskii-type inequalities, E_n behaves like $\mathcal{P}_n$, where $\mathcal{P}_n$ denotes the collection of all algebraic polynomials of degree at most n with real coefficients.

1. Introduction

Denote by $\mathcal{P}_n$ the collection of all algebraic polynomials of degree at most n with real coefficients. If we want to estimate $|p'(x)|$ for a polynomial $p \in \mathcal{P}_n$ and for a

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fixed $x \in (-1,1)$, typically we use the following inequality rather than Markov's inequality. See, for example, DeVore and Lorentz [8] or Lorentz [15].

Theorem 1.1 (Bernstein's Inequality). *The inequality*

$$|p'(x)| \leq \frac{n}{\sqrt{1-x^2}}\,\|p\|_{[-1,1]}\,, \qquad -1 < x < 1$$

holds for every $p \in \mathcal{P}_n$.

In the above theorem and throughout this paper

$$\|p\|_A := \sup_{x \in A} |p(x)|$$

for real-valued functions p defined on a set A. Exponential sums belong to one of those concrete families of functions which are the most frequently used in nonlinear approximation theory. Exponential sums arise in an approximation problem often encountered for the analysis of decay processes in natural sciences. A given empirical function on a real interval is to be approximated by sums of the form

$$\sum_{j=1}^{n} a_j e^{\lambda_j t}\,,$$

where the parameters a_j and λ_j are to be determined, while n is fixed. In [3] we proved the "right" Bernstein-type inequality for exponential sums. This inequality is the key to proving inverse theorems for approximation by exponential sums, as we will elaborate later. Let

$$E_n := \left\{ f : f(t) = a_0 + \sum_{j=1}^{n} a_j e^{\lambda_j t}\,, \quad a_j, \lambda_j \in \mathbb{R} \right\}.$$

So E_n is the collection of all $n+1$ term exponential sums with constant first term. Schmidt [22] proved that there is a constant $c(n)$ depending only on n so that

$$\|f'\|_{[a+\delta,b-\delta]} \leq c(n)\delta^{-1}\|f\|_{[a,b]}$$

for every $p \in E_n$ and $\delta \in \left(0, \frac{1}{2}(b-a)\right)$. Lorentz [16] improved Schmidt's result by showing that for every $\alpha > \frac{1}{2}$, there is a constant $c(\alpha)$ depending only on α so that $c(n)$ in the above inequality can be replaced by $c(\alpha)n^{\alpha \log n}$ (Xu improved this to allow $\alpha = \frac{1}{2}$), and he speculated that there may be an absolute constant c so that Schmidt's inequality holds with $c(n)$ replaced by cn. We [1] proved a weaker version of this conjecture with cn^3 instead of cn. Our main result in [3] shows that Schmidt's inequality holds with $c(n) = 2n-1$. This result can also be formulated as

Theorem 1.2. *We have*

$$\sup_{0 \neq f \in E_n} \frac{|f'(y)|}{\|f\|_{[a,b]}} \leq \frac{2n-1}{\min\{y-a, b-y\}}\,.$$

In this Bernstein-type inequality even the pointwise factor is sharp up to a multiplicative absolute constant. More precisely in our paper [3] the inequality

$$\frac{1}{e-1}\,\frac{n-1}{\min\{y-a,b-y\}} \leq \sup_{0\neq f\in E_n} \frac{|f'(y)|}{\|f\|_{[a,b]}}$$

is established. Theorem 1.2 follows easily from our other central result in [3], which states that the equality

$$\sup_{0\neq f\in \widetilde{E}_{2n}} \frac{|f'(0)|}{\|f\|_{[-1,1]}} = 2n-1 \tag{1.1}$$

holds, where

$$\widetilde{E}_{2n} := \left\{ f : f(t) = a_0 + \sum_{j=1}^{n} \left(a_j e^{\lambda_j t} + b_j e^{-\lambda_j t} \right), \quad a_j, b_j, \lambda_j \in \mathbb{R} \right\}.$$

Bernstein-type inequalities play an important role in approximation theory via a machinery developed by Bernstein, which turn Bernstein-type inequalities into inverse theorems of approximation. See, for example Lorentz [16] and DeVore and Lorentz [8]. Roughly speaking, our Theorem 1.2 implies that inverse theorems of approximation, over large classes of functions, by the particular exponential sums f of the form

$$f(t) = a_0 + \sum_{j=1}^{n} a_j e^{jt}, \qquad a_j \in \mathbb{R}$$

are essentially the same as those of approximation by arbitrary exponential sums f with $n+1$ terms of the form

$$f(t) = a_0 + \sum_{j=1}^{n} a_j e^{\lambda_j t}, \qquad a_j, \lambda_j \in \mathbb{R}\,.$$

So one deduces in a standard fashion, see Lorentz [16] or DeVore and Lorentz [8], for example, that if there is a sequence $(f_n)_{n=1}^{\infty}$ of exponential sums with $f_n \in E_n$ that approximates f on an interval $[a,b]$ uniformly with errors $\|f-f_n\|_{[a,b]} = o\big(n^{-m}\big)$, $m \in \mathbb{N}$, then f is m times continuously differentiable on (a,b).

The classical Remez inequality states that if p is a polynomial of degree at most n, $s \in (0,2)$, and

$$m\left(\{x \in [-1,1] : |p(x)| \leq 1\}\right) \geq 2-s\,,$$

then

$$\|p\|_{[-1,1]} \leq T_n\left(\frac{2+s}{2-s}\right),$$

where $T_n(x) = \cos(n \arccos x)$ is the Chebyshev polynomial of degree n. This inequality is sharp and

$$T_n\left(\frac{2+s}{2-s}\right) \leq \exp(5n\sqrt{s})\,, \qquad s \in (0,1]\,.$$

Remez-type inequalities turn out to be very useful in various problems of approximation theory. See, for example Borwein and Erdélyi [2], [4], and [5], Erdélyi [9], [10], and [11], Erdélyi and Nevai [12], Freud [13], and Lorentz, Golitschek, and Makovoz [17].

In this paper we establish an essentially sharp Remez-type inequality for E_n. As an application, we also prove an essentially sharp Nikolskii-type inequality for E_n. The notation

$$\|f\|_A := \sup_{x \in A} |f(x)| \qquad \text{and} \qquad \|f\|_{L_q(A)} := \left(\int_A |f|^q \right)^{1/q}$$

is used throughout this paper for measurable functions f defined on a measurable set $A \subset \mathbb{R}$ and for $q \in (0, \infty)$. The space of all real-valued continuous functions on a set $A \subset [0, \infty)$ equipped with the uniform norm is denoted by $C(A)$. The space $L_q(A)$ is defined as the collection of equivalence classes of real-valued measurable functions for which $\|f\|_{L_q(A)} < \infty$. The equivalence classes are defined by the equivalence relation $f \sim g$ if $f = g$ almost everywhere on A. When $A := [a, b]$ is a finite closed interval, we use the notation $L_q[a, b] := L_q(A)$.

2. New Results

As before, let

$$E_n := \left\{ f : f(t) = a_0 + \sum_{j=1}^{n} a_j e^{\lambda_j t}\,, \quad a_j, \lambda_j \in \mathbb{R} \right\}.$$

So E_n is the collection of all $n+1$ term exponential sums with constant first term.

We prove the following two theorems.

Theorem 2.1 (Remez-Type Inequality for E_n at 0). *Let $s \in \left(0, \frac{1}{2}\right]$. There are absolute constants $c_1 > 0$ and $c_2 > 0$ such that*

$$\exp(c_1 ns) \le \sup_f |f(0)| \le \exp(c_2 ns)\,,$$

where the supremum is taken for all $f \in E_n$ satisfying

$$m\left(\{x \in [-1, 1] : |f(x)| \le 1\}\right) \ge 2 - s\,.$$

Theorem 2.2 (Nikolskii-Type Inequality for E_n). *There are absolute constants $c_3 > 0$ and $c_4 > 0$ such that*

$$c_3^{1+1/q} \left(\frac{1 + qn}{\min\{y - a, b - y\}} \right)^{1/q} \le \sup_{0 \ne f \in E_n} \frac{|f(y)|}{\|f\|_{L_q[a,b]}} \le \left(\frac{c_4(1 + qn)}{\min\{y - a, b - y\}} \right)^{1/q}$$

for every $a < y < b$ and $q > 0$.

The above results are interesting additions to our result below proved in [3], see also [2] .

Theorem 2.3 (Bernstein-Type Inequality for E_n). *We have*

$$\frac{1}{e-1}\,\frac{n-1}{\min\{y-a,b-y\}} \leq \sup_{0\neq f\in E_n} \frac{|f'(y)|}{\|f\|_{[a,b]}} \leq \frac{2n-1}{\min\{y-a,b-y\}}$$

for every $a<y<b$

It is worth noting that in the above Remez- and Nikolskii-type inequalities E_n behaves like $\mathcal{P}_n$, where $\mathcal{P}_n$ denotes the collection of all polynomials of degree at most n with real coefficients. Note that for $0<p\leq 2$, the upper bound of Theorem 2.2 is essentially proved in [3] (cf. Theorem 3.4) and in [2] (pages 289–291). However, the methods used there cannot be extended to all $p>0$.

3. Chebyshev and Descartes Systems

The proof of our main result relies heavily on the observation that for every $0<\lambda_1<\lambda_2<\cdots<\lambda_n$,

$$(1\,, \cosh(\lambda_1 t)-1\,, \cosh(\lambda_2 t)-1\,, \ldots\,, \cosh(\lambda_n t)-1)$$

is a Descartes system on $[0,\infty)$. We will also need some simple properties of Chebyshev systems. In this section we give the definitions of Chebyshev and Descartes systems. We also formulate some of their elementary properties. The only result of this section that is not to be found in standard sources is the critical Lemma 3.5. The remaining theory can be found in [2] or [14], for example.

Definition 3.1 (Chebyshev System). Let A be a Hausdorff space. The sequence $(f_0,f_1,\ldots,f_n)$ is called a (real) Chebyshev system of dimension $n+1$ on A if $f_0,f_1,\ldots,f_n$ are real-valued continuous functions on A, $\text{span}\{f_0,f_1,\ldots,f_n\}$ over $\mathbb{R}$ is an $n+1$ dimensional subspace of $C(A)$, and any $f\in\text{span}\{f_0,f_1,\ldots,f_n\}$ that has $n+1$ distinct zeros on A is identically zero.

If $(f_0,f_1,\ldots,f_n)$ is a Chebyshev system on A, then $\text{span}\{f_0,f_1,\ldots,f_n\}$ is called a Chebyshev space or Haar space on A.

Implicit in the definition is that A contains at least $n+1$ points. Being a Chebyshev system is a property of the space spanned by the elements of the system, so every basis of a Chebyshev space is a Chebyshev system.

A point $x_0\in(a,b)$ is called a *double zero* of an $f\in C[a,b]$ if $f(x_0)=0$ and $f(x_0-\varepsilon)f(x_0+\varepsilon)>0$ for all sufficiently small $\varepsilon>0$ (in other words if f vanishes without changing sign at x_0. It is easy to see that if $\{f_0,f_1,\ldots,f_n\}$ is a Chebyshev system on $[a,b]\subset\mathbb{R}$, then then every $0\neq p\in\text{span}\{f_0,f_1,\ldots,f_n\}$ has at most n zeros even if every double zero is counted twice; see E.10 of Section 3.1 of [2].

The following simple equivalences are well known facts of linear algebra.

Proposition 3.2 (Equivalences). *Let $f_0,f_1,\ldots,f_n$ be real-valued continuous functions on a Hausdorff space A (containing at least $n+1$ points). Then the following are equivalent.*

a] *Every* $0 \neq p \in \operatorname{span}\{f_0, f_1, \dots, f_n\}$ *has at most* n *distinct zeros on* A.

b] *If* $x_0, x_1, \dots, x_n$ *are distinct elements of* A *and* $y_0, y_1, \dots, y_n$ *are real numbers then there exists a unique* $p \in \operatorname{span}\{f_0, f_1, \dots, f_n\}$ *so that*

$$p(x_i) = y_i\,, \qquad i = 1, 2, \dots, n\,.$$

c] *If* $x_0, x_1, \dots, x_n$ *are distinct points of* A, *then*

$$D(x_0, x_1, \dots, x_n) := \begin{vmatrix} f_0(x_0) & \dots & f_n(x_0) \\ \vdots & \ddots & \vdots \\ f_0(x_n) & \dots & f_n(x_n) \end{vmatrix} \neq 0\,.$$

Definition 3.3 (Descartes System). *The system* $(f_0, f_1, \dots, f_n)$ *is said to be a Descartes system (or order complete Chebyshev system) on an interval* I *if each* $f_i \in C(I)$ *and*

$$D\begin{pmatrix} f_{i_0} & f_{i_1} & \dots & f_{i_m} \\ x_0 & x_1 & \dots & x_m \end{pmatrix} > 0$$

for any $0 \le i_0 < i_1 < \dots < i_m \le n$ *and for any* $x_0 < x_1 < \dots < x_m$ *from* I. *The definition of an infinite Descates system* $(f_0, f_1, \dots)$ *on* I *is analogous.*

This is a property of the basis. It implies that any finite dimensional subspace generated by some system elements is a Chebyshev space on I. We remark the trivial fact that a Descartes system on I is a Descartes system on any subinterval of I.

Lemma 3.4. *The system*

$$(e^{\lambda_0 t}, e^{\lambda_1 t}, \dots)\,, \qquad \lambda_0 < \lambda_1 < \cdots$$

is a Descartes system on $(-\infty, \infty)$. *In particular, it is also a Chebyshev system on* $(-\infty, \infty)$.

Proof. See, for example, Karlin and Studden [14]. □

The following lemma plays a crucial role in the proof of Theorem 2.1.

Lemma 3.5. *Suppose* $0 < \lambda_1 < \lambda_2 < \cdots$. *Then*

$$(1\,, \cosh(\lambda_1 t) - 1\,, \cosh(\lambda_2 t) - 1\,, \dots)$$

is a Chebyshev system on $[0, \infty)$ *and a Descartes system on* $(0, \infty)$.

Proof. Let $0 < i_1 < i_2 < \cdots < i_m$ be fixed integers. First we show that

$$(1\,, \cosh(\lambda_{i_1} t) - 1\,, \cosh(\lambda_{i_2} t) - 1\,, \dots\,, \cosh(\lambda_{i_m} t) - 1)$$

is a Chebyshev system on $[0,\infty)$. To see this, let

$$0 \neq f \in \operatorname{span}\{1\,,\, \cosh(\lambda_{i_1}t)-1\,,\, \cosh(\lambda_{i_2}t)-1\,,\, \dots\,,\, \cosh(\lambda_{i_m}t)-1\}\,.$$

Then, with $\lambda_0 := 0\,,$

$$0 \neq f \in \operatorname{span}\{e^{\lambda_0 t}\,,\, e^{\pm\lambda_{i_1}t}\,,\, e^{\pm\lambda_{i_2}t}\,,\, \dots\,,\, e^{\pm\lambda_{i_m}t}\}$$

Here

$$\operatorname{span}\{e^{\lambda_{i_0} t}\,,\, e^{\pm\lambda_{i_1}t}\,,\, \dots\,,\, e^{\pm\lambda_{i_m}t}\}$$

is a Chebyshev system on $(-\infty,\infty)$ of dimension $2m+1$, hence f has at most $2m$ zeros in $(-\infty,\infty)$. Since f is even, it has at most m zeros in $[0,\infty)$. So the system

$$(1\,,\, \cosh(\lambda_{i_1}t)-1\,,\, \cosh(\lambda_{i_2}t)-1\,,\, \dots\,,\, \cosh(\lambda_{i_m}t)-1)\,.$$

is a Chebyshev system on $[0,\infty)$. It can be shown similarly that

$$(\cosh(\lambda_{i_1}t)-1\,,\, \cosh(\lambda_{i_2}t)-1\,,\, \dots\,,\, \cosh(\lambda_{i_m}t)-1)\,.$$

is a Chebyshev system on $(0,\infty)\,.$

Now we show that

$$(1\,,\, \cosh(\lambda_1 t)-1\,,\, \cosh(\lambda_2 t)-1\,,\, \dots)$$

is a Descartes system on $(0,\infty)\,.$ Since for every $0 < i_1 < i_2 < \cdots < i_m$,

$$(1\,,\, \cosh(\lambda_{i_1}t)-1\,,\, \cosh(\lambda_{i_2}t)-1\,,\, \dots\,,\, \cosh(\lambda_{i_m}t)-1)$$

is a Chebyshev system on $(0,\infty)\,,$ Proposition 3.2 implies that the determinant

$$D\begin{pmatrix} 1 & \cosh(\lambda_{i_1}t)-1 & \cosh(\lambda_{i_2}t)-1 & \dots & \cosh(\lambda_{i_m}t)-1 \\ x_0 & x_1 & x_2 & \dots & x_m \end{pmatrix}$$
$$:= \begin{vmatrix} 1 & \cosh(\lambda_{i_1}x_0)-1 & \cosh(\lambda_{i_2}x_0)-1 & \dots & \cosh(\lambda_{i_m}x_0)-1 \\ 1 & \cosh(\lambda_{i_1}x_1)-1 & \cosh(\lambda_{i_2}x_1)-1 & \dots & \cosh(\lambda_{i_m}x_1)-1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \cosh(\lambda_{i_1}x_m)-1 & \cosh(\lambda_{i_2}x_m)-1 & \dots & \cosh(\lambda_{i_m}x_m)-1 \end{vmatrix}$$

is non-zero for any $0 < x_0 < x_1 < \cdots < x_m < \infty\,.$ So it only remains to prove that it is positive whenever $0 < x_0 < x_1 < \cdots < x_m < \infty$. Now let

$$D(\alpha) := D\begin{pmatrix} 1 & (\cosh\lambda_{i_1}t)-1 & \cosh(\lambda_{i_2}t)-1 & \dots & \cosh(\lambda_{i_m}t)-1 \\ \alpha x_0 & \alpha x_1 & \alpha x_2 & \dots & \alpha x_m \end{pmatrix}$$
$$:= \begin{vmatrix} 1 & \cosh(\lambda_{i_1}\alpha x_0)-1 & \cosh(\lambda_{i_2}\alpha x_0)-1 & \dots & \cosh(\lambda_{i_m}\alpha x_0)-1 \\ 1 & \cosh(\lambda_{i_1}\alpha x_1)-1 & \cosh(\lambda_{i_2}\alpha x_1)-1 & \dots & \cosh(\lambda_{i_m}\alpha x_1)-1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \cosh(\lambda_{i_1}\alpha x_m)-1 & \cosh(\lambda_{i_2}\alpha x_m)-1 & \dots & \cosh(\lambda_{i_m}\alpha x_m)-1 \end{vmatrix}$$

and, with $\lambda_0 := 0$, let

$$D^*(\alpha) := D\begin{pmatrix} e^{\lambda_{i_0}t} & \frac{1}{2}e^{\lambda_{i_1}t} & \dots & \frac{1}{2}e^{\lambda_{i_m}t} \\ \alpha x_0 & \alpha x_1 & \dots & \alpha x_m \end{pmatrix}$$

$$:= \begin{vmatrix} e^{\lambda_{i_0}\alpha x_0} & \frac{1}{2}e^{\lambda_{i_1}\alpha x_0} & \dots & \frac{1}{2}e^{\lambda_{i_m}\alpha x_0} \\ e^{\lambda_{i_0}\alpha x_1} & \frac{1}{2}e^{\lambda_{i_1}\alpha x_1} & \dots & \frac{1}{2}e^{\lambda_{i_m}\alpha x_1} \\ \vdots & \vdots & \ddots & \vdots \\ e^{\lambda_{i_0}\alpha x_m} & \frac{1}{2}e^{\lambda_{i_1}\alpha x_m} & \dots & \frac{1}{2}e^{\lambda_{i_m}\alpha x_m} \end{vmatrix},$$

where $0 < x_0 < x_1 < \dots < x_m < \infty$ are fixed. Since

$$(1\,,\, \cosh(\lambda_{i_1}t) - 1\,,\, \cosh(\lambda_{i_2}t) - 1\,,\, \dots\,,\, \cosh(\lambda_{i_m}t) - 1)$$

and

$$(e^{\lambda_{i_0}t}\,,\, e^{\lambda_{i_1}t}\,,\, \dots\,,\, e^{\lambda_{i_m}t})$$

are Chebyshev systems on $(0, \infty)$, $D(\alpha)$ and $D^*(\alpha)$ are continuous non-vanishing functions of α on $(0, \infty)$. Now observe that

$$\lim_{\alpha\to\infty} |D(\alpha)| = \lim_{\alpha\to\infty} |D^*(\alpha)| = \infty \qquad \text{and} \qquad \lim_{\alpha\to\infty} \frac{D(\alpha)}{D^*(\alpha)} = 1\,.$$

Since

$$(e^{\lambda_{i_0}t}\,,\, e^{\lambda_{i_1}t}\,,\, \dots\,,\, e^{\lambda_{i_m}t})$$

is a Descartes system on $(-\infty, \infty)$, $D^*(\alpha) > 0$ for every $\alpha > 0$. So the above limit relations imply that $D(\alpha) > 0$ for every sufficiently large $\alpha > 0$, hence for every $\alpha > 0$. In particular,

$$D(1) = D\begin{pmatrix} 1 & \cosh(\lambda_{i_1}t) - 1 & \cosh(\lambda_{i_2}t) - 1 & \dots & \cosh(\lambda_{i_m}t) - 1 \\ x_0 & x_1 & x_2 & \dots & x_m \end{pmatrix} > 0\,.$$

It can be shown similarly that

$$D\begin{pmatrix} \cosh(\lambda_{i_1}t) - 1 & \cosh(\lambda_{i_2}t) - 1 & \dots & \cosh(\lambda_{i_m}t) - 1 \\ x_1 & x_2 & \dots & x_m \end{pmatrix}$$

$$:= \begin{vmatrix} \cosh(\lambda_{i_1}x_0) - 1 & \cosh(\lambda_{i_2}x_0) - 1 & \dots & \cosh(\lambda_{i_m}x_0) - 1 \\ \cosh(\lambda_{i_1}x_1) - 1 & \cosh(\lambda_{i_2}x_1) - 1 & \dots & \cosh(\lambda_{i_m}x_1) - 1 \\ \vdots & \vdots & \ddots & \vdots \\ \cosh(\lambda_{i_1}x_m) - 1 & \cosh(\lambda_{i_2}x_m) - 1 & \dots & \cosh(\lambda_{i_m}x_m) - 1 \end{vmatrix} > 0$$

for all $0 < x_0 < x_1 < \dots < x_m < \infty$. Hence

$$(1\,,\, \cosh(\lambda_1 t) - 1\,,\, \cosh(\lambda_2 t) - 1\,,\, \dots)$$

is a Descartes system on $(0, \infty)$, indeed. $\square$

4. Chebyshev Polynomials

Throughout this paper $\Lambda := (\lambda_i)_{i=0}^{\infty}$ denotes a sequence of real numbers satisfying

$$0 < \lambda_1 < \lambda_2 < \cdots .$$

The system

$$(1\,,\,\cosh(\lambda_1 t) - 1\,,\,\cosh(\lambda_2 t) - 1\,,\,\dots\,,\,\cosh(\lambda_n t) - 1)$$

is called a (finite) cosh system. The linear space

$$H_n(\Lambda) := \text{span}\{1\,,\,\cosh(\lambda_1 t) - 1\,,\,\cosh(\lambda_2 t) - 1\,,\,\dots\,,\,\cosh(\lambda_n t) - 1\}$$

over $\mathbb{R}$ is called a (finite) cosh space. That is, the cosh space $H_n(\Lambda)$ is the collection of all possible linear combinations

$$p(t) = a_0 + \sum_{j=0}^{n} a_j(\cosh(\lambda_j t) - 1)\,, \qquad a_j \in \mathbb{R}\,.$$

The set

$$H(\Lambda) := \bigcup_{n=0}^{\infty} H_n(\Lambda) = \text{span}\{1\,,\,\cosh(\lambda_1 t) - 1\,,\,\cosh(\lambda_2 t) - 1\,,\,\dots\}$$

is called the (infinite) cosh space associated with Λ.

As we have seen in the previous section, one of the most basic properties of a cosh space $H_n(\Lambda)$ is the fact that it is a Chebyshev space on every $A \subset [0,\infty)$ containing at least $n+1$ points. That is, $H(\Lambda) \subset C(A)$ and every $p \in H_n(\Lambda)$ having at least $n+1$ (distinct) zeros in A is identically 0 on A. In fact, cosh spaces $H_n(\Lambda)$ are simple examples for Chebyshev spaces, hence they share the following well known properties of general Chebyshev spaces (see, for example, [2], [14], and [21]).

Theorem 4.1 (Existence of Chebyshev Polynomials). *Let A be a compact subset of $[0,\infty)$ containing at least $n+1$ points. Then there exists a unique (extended) Chebyshev polynomial*

$$T_n := T_n\{\lambda_1\,,\,\lambda_2\,,\,\dots\,,\,\lambda_n\,;A\}$$

for $H_n(\Lambda)$ on A defined by

$$T_n(x) = c\left((\cosh(\lambda_n t) - 1) - \left(a_0 + \sum_{j=1}^{n-1} a_j(\cosh(\lambda_j t) - 1)\right)\right)\,,$$

where the numbers $a_0, a_1, \dots, a_{n-1} \in \mathbb{R}$ are chosen to minimize

$$\left\|(\cosh(\lambda_n t) - 1) - \left(a_0 + \sum_{j=1}^{n-1} a_j(\cosh(\lambda_j t) - 1)\right)\right\|_A\,,$$

and where $c \in \mathbb{R}$ is a normalization constant chosen so that

$$\|T_n\|_A = 1$$

and the sign of c is determined by

$$T_n(\max A) > 0\,.$$

Theorem 4.2 (Alternation Characterization). *The Chebyshev polynomial*

$$T_n := T_n\{\lambda_1\,,\,\lambda_2\,,\,\dots\,,\,\lambda_n\,;A\} \in H_n(\Lambda)$$

is uniquely characterized by the existence of an alternation set

$$\{x_0 < x_1 < \cdots < x_n\} \subset A$$

for which

$$T_n(x_j) = (-1)^{n-j} = (-1)^{n-j}\|T_n\|_A\,, \qquad j = 0, 1, \dots, n\,.$$

5. Comparison Lemmas

In this section we establish some comparison theorems by utilizing the fact that a cosh system

$$(1\,,\,\cosh(\lambda_1 t) - 1\,, \cosh(\lambda_2 t) - 1\,, \dots\,,\,\cosh(\lambda_n t) - 1)\,, \qquad 0 < \lambda_1 < \cdots < \lambda_n\,,$$

is a Descartes system on $(0, \infty)$; see Theorem 3.5. The following comparison lemma, due to Smith [23], is valid for every Descartes system.

Lemma 5.1. *Suppose* $(f_0, f_1, \dots, f_n)$ *is a Descartes system on* $[a, b]$*. Suppose*

$$p = f_\alpha + \sum_{i=1}^{k} a_i f_{\mu_i}\,, \qquad a_i \in \mathbb{R}\,,$$

$$q = f_\alpha + \sum_{i=1}^{k} b_i f_{\nu_i}, \qquad b_i \in \mathbb{R}\,,$$

where $0 \le \mu_1 < \mu_2 < \cdots < \mu_k \le n\,,\ \ 0 \le \nu_1 < \nu_2 < \cdots < \nu_k \le n\,,$

$$0 \le \nu_i \le \mu_i < \alpha\,, \qquad i = 1, 2, \dots, m\,,$$

and

$$\alpha < \mu_i \le \nu_i \le n\,, \qquad i = m+1, m+2, \dots, k$$

with strict inequality for at least one index $i = 1, 2, \dots, k$*. Then*

$$p(x_i) = q(x_i) = 0\,, \qquad i = 1, 2, \dots, k\,,$$

with distinct $x_i \in [a, b]$ *implies*

$$|p(x)| \le |q(x)|$$

for every $x \in [a, b]$ *with strict inequality for every*

$$x \in [a, b] \setminus \{x_1, x_2, \dots, x_k\}\,.$$

To formulate the next lemmas we introduce the following notation. Let

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n\,, \quad 0 < \gamma_1 < \gamma_2 < \cdots < \gamma_n$$

and

$$\gamma_i \leq \lambda_i\,, \qquad i = 1, 2, \ldots, n\,.$$

Let

$$H_n(\Lambda) := \operatorname{span}\{(1\,, \cosh(\lambda_1 t) - 1\,, \cosh(\lambda_2 t) - 1\,, \ldots\,, \cosh(\lambda_n t) - 1)\}$$

and

$$H_n(\Gamma) := \operatorname{span}\{(1\,, \cosh(\gamma_1 t) - 1\,, \cosh(\gamma_2 t) - 1\,, \ldots\,, \cosh(\gamma_n t) - 1)\}\,.$$

Let $A \subset (0, \infty)$ be a compact set containing at least $n+1$ points. Let

$$T_{n,\lambda} := T_n\{\lambda_1\,, \lambda_2\,, \ldots\,, \lambda_n\,; A\}$$

and

$$T_{n,\gamma} := T_n\{\gamma_1\,, \gamma_2\,, \ldots\,, \gamma_n\,; A\}$$

denote the Chebyshev polynomials on A for $H_n(\Lambda)$ and $H_n(\Gamma)$, respectively; see Theorems 4.1 and 4.2.

Lemma 5.2. *Let $A \subset (0, \infty)$ be a compact set containing at least $n+1$ points. Then*

$$\max_{0 \neq p \in H_n(\Lambda)} \frac{|p(0)|}{\|p\|_A}$$

is attained by $p = T_{n,\lambda}$.

Proof. A simple compactness argument shows that the maximum in the lemma is attained by some $p^* \in H_n(\Lambda)$, which can be identified as $T_{n,\lambda}$ by a standard variational method. See, for example, [14, page 295] or [21, page 101] where arguments of this variety are given. $\square$

Lemma 5.3. *Let $A \subset (0, \infty)$ be a compact set containing at least $n+1$ points. We have*

$$|T_{n,\lambda}(0)| \leq |T_{n,\gamma}(0)|\,.$$

Proof. Let $p \in H_n(\Gamma)$ interpolate $T_{n,\lambda}$ at the n zeros of $T_{n,\lambda}$ in $(0, \infty)$ and at 0. It follows from Lemma 5.1 that

$$|p(x)| \leq |T_{n,\lambda}(x)|\,, \qquad x \in [0, \infty)\,.$$

In particular,

$$\|p\|_A \leq \|T_{n,\lambda}\|_A = 1\,,$$

which, together with $p(0) = T_{n,\lambda}(0)$ and Lemma 5.2 gives

$$|T_{n,\lambda}(0)| = |p(0)| \leq \frac{|p(0)|}{\|p\|_A} \leq \frac{|T_{n,\gamma}(0)|}{\|T_{n,\gamma}\|_A} = |T_{n,\gamma}(0)|\,.$$

This proves the lemma. $\square$

The main result of this section is the following lemma. It plays a crucial role in the proof of Theorem 2.1

Lemma 5.4. *Let $A \subset (0,\infty)$ be a compact set containing at least $n+1$ points. We have*

$$\max_{0 \neq p \in H_n(\Lambda)} \frac{|p(0)|}{\|p\|_A} \leq \max_{0 \neq p \in H_n(\Gamma)} \frac{|p(0)|}{\|p\|_A}\,.$$

Proof. Combining Lemmas 5.2 and 5.3, we obtain that

$$\begin{aligned}\max_{0 \neq p \in H_n(\Lambda)} \frac{|p(0)|}{\|p\|_A} &= \frac{|T_{n,\lambda}(0)|}{\|T_{n,\lambda}\|_A} = |T_{n,\lambda}(0)| \leq |T_{n,\gamma}(0)| \\ &= \frac{|T_{n,\gamma}(0)|}{\|T_{n,\gamma}\|_A} \leq \max_{0 \neq p \in H_n(\Gamma)} \frac{|p(0)|}{\|p\|_A}\,,\end{aligned}$$

which implies the inequality of the lemma. □

6. On the span of $\{1\,, \cosh(\varepsilon t) - 1\,, \cosh(2\varepsilon t) - 1\,, \dots, \cosh(n\varepsilon t) - 1\}$

In this section we study the space

$$H_n(\varepsilon) := \text{span}\{1, \cosh \varepsilon t, \cosh 2\varepsilon t, \dots, \cosh n\varepsilon t\}\,,$$

where $\varepsilon > 0$ is fixed. Observe that every $f \in H_n(\varepsilon)$ is of the form

$$f(t) = Q(\cosh \varepsilon t)\,, \qquad Q \in \mathcal{P}_n\,. \tag{6.1}$$

For $n \in \mathbb{N}$, $\epsilon > 0$, and $s \in (0,1)$, let

$$A_{n,\varepsilon,s} := \{f \in H_n(\varepsilon) : \; m\left(\{t \in [0,1] : |f(t)| \leq 1\}\right) \geq 1 - s\}\,,$$

and choose an extremal element

$$f^* = f^*_{n,\varepsilon,s} \in A_{n,\varepsilon,s}$$

such that

$$|f^*(0)| = \sup\{|f(0)| : \; f \in A_{n,\varepsilon,s}\}\,.$$

The existence of such an extremal element follows easily from the observation that $A_{n,\varepsilon,s}$ is a closed and bounded, hence compact subset of $H_n(\varepsilon)$ in the uniform (hence in any) norm on $[-1,1]$, and we omit the details. We introduce $Q^* = Q^*_{n,\varepsilon,s} \in \mathcal{P}_n$ by

$$f^*(t) = Q^*(\cosh(\varepsilon t))\,.$$

Now we shall study the properties of f^* and Q^*.

Proposition 6.1. *The polynomial Q^* has only real zeros.*

Proof. Suppose that Q^* vanishes at a nonreal $z_0 \in \mathbb{C}$. Then

$$R_{n,\epsilon,\eta_1,\eta_2}(t) := (1+\eta_1) Q^*(\cosh(\varepsilon t)) \left(1 - \frac{\eta_2 (\cosh(\varepsilon t) - 1)^2}{(\cosh(\varepsilon t) - z_0)(\cosh(\varepsilon t) - \overline{z}_0)}\right)$$

with sufficiently small $\eta_1 > 0$ and $\eta_2 > 0$ is in $A_{n,\varepsilon,s}$ and contradicts the extremality of f^*. Hence the proposition is proved. □

Proposition 6.2. *The polynomial* Q^* *has each of its zeros in* $[1, \cosh\varepsilon]$.

Proof. Suppose that Q^* vanishes at a $z_0 \in \mathbb{C}$ outside $[1, \cosh\varepsilon]$. By Proposition 6.1 we may assume that z_0 is real, hence $z_0 \in \mathbb{R} \setminus [1, \cosh\varepsilon]$. If $z_0 \in (-\infty, 1)$, then

$$R_{n,\epsilon,\eta_1,\eta_2}(t) := (1+\eta_1)Q^*(\cosh(\varepsilon t))\left(1 - \frac{\eta_2(\cosh(\varepsilon t)-1)}{(\cosh(\varepsilon t)-z_0)}\right)$$

with sufficiently small $\eta_1 > 0$ and $\eta_2 > 0$ is in $A_{n,\varepsilon,s}$ and contradicts the extremality of f^*. If $z_0 \in (\cosh\varepsilon, \infty)$, then

$$R_{n,\epsilon,\eta_1,\eta_2}(t) := (1+\eta_1)Q^*(\cosh(\varepsilon t))\left(1 - \frac{\eta_2(\cosh(\varepsilon t)-1)}{(z_0-\cosh(\varepsilon t))}\right)$$

with sufficiently small $\eta_1 > 0$ and $\eta_2 > 0$ is in $A_{n,\varepsilon,s}$ and contradicts the extremality of f^*. Hence the proposition is proved. □

For functions f defined on an interval I, we introduce the notation

$$M(f, I) = \{t \in I : |f(t)| \le 1\}\ .$$

Obviously, for a function $f \in H_n(\varepsilon)$, the set $M(f, [0,1])$ comprises at most n closed intervals possessing no common points, otherwise repeated applications of Rolle's Theorem would imply that

$$0 \ne f' \in \operatorname{span}\{e^{\pm\varepsilon t}, e^{\pm 2\varepsilon t}, \dots, e^{\pm n\varepsilon t}\}$$

has at least $2n$ zeros, which is impossible by Lemma 3.4. These intervals will be called the portions of $M(f, [0,1])$. A function $f \in H_n(\varepsilon)$ has a representation (6.1). Then the set $M(Q, [1, \cosh\varepsilon])$ comprises at most n closed intervals possessing no common points, otherwise repeated applications of Rolle's Theorem would imply that $0 \ne Q' \in \mathcal{P}_{n-1}$ has at least n zeros in $(1, \cosh\varepsilon)$, which is impossible. These intervals will be called the portions of $M(Q, [1, \cosh\varepsilon])$.

Proposition 6.3. *Every portion of* $M(Q^*, [1, \cosh\varepsilon])$ *contains at least one zero of* Q^*.

Proof. Without loss of generality we may assume that $n \ge 2$. Suppose that there is a portion of $M(Q^*, [1, \cosh\varepsilon])$ with no zeros in it. Then, using Rolle's Theorem, we would be able to find two zeros of of $Q^{*\prime}$ so that Q^* has no zeros between them. This contradicts Proposition 6.1. □

Proposition 6.4. *There is only one portion of* $M(Q^*, [1, \cosh\varepsilon])$.

Proof. Suppose that there are at least two different portions of $M(Q^*, [1, \cosh\varepsilon])$. Because of the extremality of Q^*, we have $1 \notin M(Q^*, [1, \cosh\varepsilon])$. Let

$$[a, b] \quad (1 < a < b < \cosh\varepsilon)$$

be the closest portion of $M(Q^*, [1, \cosh\varepsilon])$ to 1. Denote by

$$(a <)x_1 \le x_2 \le \cdots \le x_m (< b)$$

the zeros of Q^* lying in $[a, b]$, where we list each zero as many times as its multiplicity, and where $m \ge 1$ by Proposition 6.3. We study

$$G_h(t) = G_{h,\varepsilon}(t) := Q^*(\cosh(\varepsilon t)) h^{-m} \frac{\prod_{j=1}^m (\cosh(\varepsilon t) - (1 + h(x_j - 1)))}{\prod_{j=1}^m (\cosh(\varepsilon t) - x_j)}\,.$$

Obviously

$$|G_h(0)| = |Q^*(1)| = |f^*(0)| \tag{6.2}$$

If $h > 1$ is sufficiently close to 1, then for every t in any portion of $M(f^*, [0,1])$ different from $[\varepsilon^{-1}\cosh^{-1} a, \varepsilon^{-1}\cosh^{-1} b]$, we have

$$|G_h(t)| \le |Q^*(\cosh(\varepsilon t))| = |f^*(t)| \le 1\,. \tag{6.3}$$

Furthermore, it is easy to check that instead of the portion

$$[\varepsilon^{-1}\cosh^{-1} a\,,\, \varepsilon^{-1}\cosh^{-1} b]$$

of $M(f^*, [0,1])$, $M(G_h, [0,1])$ has a portion containing the interval

$$I := [\varepsilon^{-1}\cosh^{-1}(1 + h(a-1))\,,\, \varepsilon^{-1}\cosh^{-1}(1 + h(b-1))]\,,$$

assuming that $h > 1$ is sufficiently close to 1. Indeed, this follows from the fact that

$$|G_h(\varepsilon^{-1}\cosh^{-1}(1 + h(\cosh(\varepsilon t) - 1)))| \le |f^*(t)| \le 1$$

for every $t \in [\varepsilon^{-1}\cosh^{-1} a, \varepsilon^{-1}\cosh^{-1} b]$, assuming that $h > 1$ is sufficiently close to 1.

A simple calculation shows that

$$\begin{aligned}
&\frac{d}{dh}[\varepsilon^{-1}\cosh^{-1}(1 + h(b-1)) - \varepsilon^{-1}\cosh^{-1}(1 + h(a-1))]\Big|_{h=1} \\
&\qquad = \frac{\varepsilon^{-1}(b-1)}{\sqrt{(1 + h(b-1))^2 - 1}} - \frac{\varepsilon^{-1}(a-1)}{\sqrt{(1 + h(a-1))^2 - 1}}\Bigg|_{h=1} \\
&\qquad = \varepsilon^{-1}\sqrt{\frac{b-1}{b+1}} - \varepsilon^{-1}\sqrt{\frac{a-1}{a+1}} > 0\,.
\end{aligned}$$

From this we can deduce that the portion of $M(G_h, [0,1])$ containing I has larger measure than the portion $[\varepsilon^{-1}\cosh^{-1} a, \varepsilon^{-1}\cosh^{-1} b]$ of $M(f^*, [0,1])$, assuming that $h > 1$ is sufficiently close to 1. This, together with (6.3), gives that $G_h \in A_{n,\varepsilon,s'}$ with some $0 < s' < s$ if $h > 1$ is sufficiently close to 1. Therefore the functions

$$G_{h,\varepsilon,\eta}(t) := G_h(t) \pm \eta\cosh(\varepsilon t)$$

with sufficiently small $\eta > 0$ is in $A_{n,\varepsilon,s}$, and by (6.2) one of them contradicts the maximality of $f^*(t) = Q^*(\cosh(\varepsilon t))$. Thus $M(Q^*, [1, \cosh\varepsilon])$ has only one portion, indeed. $\square$

Proposition 6.5. *The only portion of* $M(Q^*, [1, \cosh\varepsilon])$ *is* $[\cosh(\varepsilon s), \cosh\varepsilon]$.

Proof. The proof is a straightforward modification of the previous proof. □

Proposition 6.6. *Let* $\varepsilon \in \left(0, \frac{1}{2}\right)$ *and* $s \in \left(0, \frac{1}{2}\right]$. *Then*

$$|f^*(0)| = |Q^*(1)| \le \exp(15ns) \ .$$

To see the above proposition we need the numerical version of Chebyshev's Inequality for polynomials. This can be formulated as follows; see, for example, [2, page 393, Theorem A.4.1].

Proposition 6.7. *Let* $[a, b]$ *be an interval, and let* $a - \frac{b-a}{2} \le y \le a$. *Then*

$$|p(y)| \le \exp\left(5n\sqrt{\frac{2(a-y)}{b-a}}\right) \|p\|_{[a,b]}$$

for every $p \in \mathcal{P}_n$.

Proof of Proposition 6.6. Using Proposition 6.7, $\varepsilon \in \left(0, \frac{1}{2}\right)$, and $s \in \left(0, \frac{1}{2}\right]$, we can estimate as

$$\begin{aligned} |f^*(0)| =& |Q^*(1)| \le \exp\left(5n\sqrt{\frac{2(\cosh(\varepsilon s)-1)}{\cosh\varepsilon - \cosh(\varepsilon s)}}\right) \\ \le& \exp\left(5n\sqrt{\frac{4(\varepsilon s)^2}{\frac{1}{2}(\cosh\varepsilon - 1)}}\right) \le \exp\left(5n\sqrt{\frac{4(\varepsilon s)^2}{\frac{1}{2}\varepsilon^2}}\right) \le \exp\left(5n\sqrt{8}s\right) \\ \le& \exp(15ns) \ . \qquad \square \end{aligned}$$

Using the extremal property of $f^*(t) = Q^*(\cosh(\varepsilon t))$, we obtain

Proposition 6.8. *Let* $\varepsilon \in \left(0, \frac{1}{2}\right)$ *and* $s \in \left(0, \frac{1}{2}\right]$. *Assume that* $A \subset [0,1]$ *is a compact set with Lebesgue measure* $m(A) \ge 1 - s$. *Then*

$$|f(0)| \le \exp(15ns))\,\|f\|_A$$

for all $f \in H_n(\varepsilon)$.

Combining Lemma 5.4 and Proposition 6.8, we are lead to the following

Proposition 6.9. *Let* $s \in \left(0, \frac{1}{2}\right]$. *Assume that* $A \subset [0,1]$ *is a compact set with Lebesgue measure* $m(A) \ge 1 - s$. *Let* $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n$. *Then, for all sufficiently small* $\varepsilon > 0$, *we have*

$$\max_{0 \ne p \in H_n(\Lambda)} \frac{|p(0)|}{\|p\|_A} \le \max_{0 \ne p \in H_n(\varepsilon)} \frac{|p(0)|}{\|p\|_A} \le \exp(15ns) \ ,$$

where, as before,

$$H_n(\Lambda) = \text{span}\{(1\,, \cosh(\lambda_1 t) - 1\,, \cosh(\lambda_2 t) - 1\,, \ldots\,, \cosh(\lambda_n t) - 1)\}\,.$$

7. Proof of Theorem 2.1

Using Sections 4, 5, and 6, we can easily prove Theorem 2.1.

Proof of Theorem 2.1. First we prove the upper bound. Let $s \in (0, \frac{1}{2}]$. Assume that $f \in E_n$ and

$$m(\{t \in [-1,1] : |f(t)| \leq 1\}) \geq 2 - s\,.$$

Then

$$g(t) = \frac{1}{2}(f(t) + f(-t)) \in H_n(\Lambda)\,,$$

where, as before,

$$H_n(\Lambda) = \text{span}\{(1\,, \cosh(\lambda_1 t) - 1\,, \cosh(\lambda_2 t) - 1\,, \dots\,, \cosh(\lambda_n t) - 1)\}$$

with some $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n$. We have $|f(0)| = |g(0)|$ and

$$m(\{t \in [0,1] : |g(t)| \leq 1\}) \geq 1 - s\,.$$

Let

$$A := \{t \in [0,1] : |g(t)| \leq 1\}\,.$$

Proposition 6.9 yields that

$$\begin{aligned} |f(0)| = &|g(0)| \leq \frac{|g(0)|}{\|g\|_A} \\ &\leq \max_{0 \neq p \in H_n(\Lambda)} \frac{|p(0)|}{\|p\|_A} \leq \max_{0 \neq p \in H_n(\varepsilon)} \frac{|p(0)|}{\|p\|_A} \leq \exp(15ns) \end{aligned}$$

for all sufficiently small $\varepsilon > 0$, which finishes the proof of the upper bound.

To prove the lower bound of the theorem, let $T_n(x) = \cos(n \arccos x)$, $x \in [-1,1]$, be the Chebyshev polynomial of degree n. Let $T_{n,s}$ be the Chebyshev polynomial T_n transformed linearly from $[-1,1]$ to $[\cosh(s/2), \cosh 1]$, that is

$$T_{n,s}(x) = T_n\left(\frac{2x}{\cosh 1 - \cosh(s/2)} - \frac{\cosh 1 + \cosh(s/2)}{\cosh 1 - \cosh(s/2)}\right)\,.$$

Let $S_n \in E_{2n}$ be defined by

$$S_n(t) := T_{n,s}(\cosh t)\,.$$

Then

$$m\left(\{t \in [-1,1] : |S_n(t)| \leq 1\}\right) = 2 - 2\frac{s}{2} = 2 - s$$

and

$$|S_n(0)| = |T_{n,s}(\cosh 0)| = |T_{n,s}(1)| \geq \exp\left(c_1 n \sqrt{\cosh(s/2) - 1}\right) \geq \exp(c_2 ns)$$

with absolute constants $c_1 > 0$ and $c_2 > 0$. Here we used $s \in (0, \frac{1}{2}]$ and the explicit formula

$$T_n(x) = \frac{1}{2}\left(\left(x + \sqrt{x^2-1}\right)^n + \left(x - \sqrt{x^2-1}\right)^n\right)\,, \qquad x \in \mathbb{R} \setminus (-1,1)\,.$$

This finishes the proof of the lower bound of the theorem. $\square$

Proof of Theorem 2.2. We show that the upper bound in Theorem 2.1 implies the upper bound in Theorem 2.2. Let $f \in E_n$. Then

$$m\left(\left\{t \in [-1,1] : |f(t)|^q \geq \frac{1}{2}|f(0)|^q\right\}\right) \geq \frac{c}{1+qn} \tag{7.1}$$

with a sufficiently small $c > 0$. To see this suppose to the contrary that with an absolute constant $0 < c < \frac{1}{2}$ we have

$$m\left(\left\{t \in [-1,1] : |f(t)|^q \geq \frac{1}{2}|f(0)|^q\right\}\right) < \frac{c}{1+qn}\,.$$

Then, with $g(t) := 2^{1/q}|f(0)|^{-1}f(t)$, we have $g \in E_n$, $g(0) = 2^{1/q}$ and

$$\begin{aligned} m\left(\{t \in [-1,1] : |g(t)| \leq 1\}\right) &= m\left(\left\{t \in [-1,1] : |f(t)|^q \leq \frac{1}{2}|f(0)|^q\right\}\right) \\ &\geq 2 - \frac{c}{1+qn}\,. \end{aligned}$$

Hence by Theorem 2.1

$$2^{1/q} \leq \exp\left(\frac{c_2 cn}{1+qn}\right)\,,$$

that is

$$2 \leq \exp\left(\frac{c_2 cqn}{1+qn}\right) \leq e^{c_2 c}\,,$$

which is impossible if $c > 0$ is sufficiently small. So (7.1) holds, indeed.

Integrating only on the set

$$I := \left\{t \in [-1,1] : |f(t)|^q \geq \frac{1}{2}|f(0)|^q\right)\,,$$

we obtain that

$$\begin{aligned} \|f\|_{L_q[-1,1]} &= \left(\int_{-1}^{1} |f(t)|^q\,dt\right)^{1/q} \geq \left(\int_I |f(t)|^q\,dt\right)^{1/q} \\ &\geq \left(\frac{c}{1+qn}\frac{1}{2}|f(0)|^q\right)^{1/q} = \left(\frac{c}{2(1+qn)}\right)^{1/q} |f(0)|\,. \end{aligned}$$

So

$$|f(0)| \leq \left(\frac{2}{c}(1+qn)\right)^{1/q} \|f\|_{L_q[-1,1]}\,.$$

Finally it follows by a linear transformation that

$$\|f\|_{[a+\delta,b-\delta]} \leq \left(\frac{2}{c\delta}(1+qn)\right)^{1/q} \|f\|_{L_q[a,b]}$$

for every $f \in E_n$, $a < b$, and $0 < \delta \leq \frac{b-a}{2}$. Hence the upper bound in Theorem 2.2 is proved.

To prove the lower bound of the theorem we proceed as follows. It follows from [2] (E.19 a] on page 413) that for every $n \in \mathbb{N}$ and $q \in (0, \infty)$ there are real algebraic polynomials $h_{n,q} \in \mathcal{P}_{\lfloor n/2 \rfloor}$ (where $\mathcal{P}_n$ denotes the set of all real algebraic polynomials of degree at most n) such that

$$|h_{n,q}(0)| \geq c_5^{1+1/q}(1+qn)^{2/q}\|h_{n,q}\|_{L_q[0,1]}$$

with an absolute constant $c_5 > 0$. Then, with the help of Theorems A.4.1 and A.4.4 of [2] (pages 393 and 395, respectively), we can easily deduce that the polynomials $p_{n,q} \in \mathcal{P}_n$ defined by

$$p_{n,q}(x) := h_{n,q}(x^2)$$

satisfy

$$|p_{n,q}(0)| \geq c_6^{1+1/q}(1+qn)^{1/q}\|p_{n,q}\|_{[-1,1]}$$

with an absolute constant $c_6 > 0$. Using the substitution $y = x + 1$ we obtain that the polynomials $P_{n,q} \in \mathcal{P}_n$ defined by

$$P_{n,q}(x) := p_{n,q}(x+1)$$

satisfy

$$|P_{n,q}(1)| \geq c_7^{1+1/q}(1+qn)^q \left(\int_0^2 |P_{n,q}(y)|^q \, dy \right)^{1/q}$$

with an absolute constant $c_7 > 0$. Using the substitution $y = e^t$, we deduce that the exponential sums $R_{n,q} \in E_{n+1}$ defined by

$$R_{n,q}(t) := P_{n,q}(e^t)e^{t/q}$$

satisfy

$$|R_{n,q}(0)| \geq c_7^{1+1/q}(1+qn)^{1/q} \left(\int_{-\infty}^{\log 2} |R_{n,q}(t)|^q \, dt \right)^{1/q} .$$

Using the substitution

$$u = \frac{\delta t}{\log 2} + b - \delta \, ,$$

we obtain that the exponential sums $S_{n,q,\delta,b} \in E_{n+1}$ defined by

$$S_{n,q,\delta,b}(u) := R_{n,q}\left(\frac{\delta t}{\log 2} + b - \delta \right)$$

satisfy

$$|S_{n,q,\delta,b}(b-\delta)| \geq c_8^{1+1/q} \left(\frac{1+qn}{\delta} \right)^{1/q} \left(\int_{-\infty}^{b} |S_{n,q,\delta,b}(u)|^q \, du \right)^{1/q}$$

with an absolute constant $c_8 > 0$. Finally observe that the exponential sums $U_{n,q,\delta,a} \in E_{n+1}$ defined by

$$U_{n,q,\delta,a}(u) := R_{n,q}\left(\frac{-\delta t}{\log 2} + a + \delta\right)$$

satisfy

$$|U_{n,q,\delta,a}(a+\delta)| \geq c_8^{1+1/q}\left(\frac{1+qn}{\delta}\right)^{1/q}\left(\int_a^\infty |U_{n,q,\delta,a}(u)|^q\,du\right)^{1/q}.$$

This finishes the proof of the lower bound of the theorem. □

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