INEQUALITIES FOR EXPONENTIAL SUMS

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November 8, 2016

Abstract. We study the classes

$$\mathcal{E}_n := \left\{ f : f(t) = \sum_{j=1}^{n} a_j e^{\lambda_j t}\,, \quad a_j, \lambda_j \in \mathbb{C} \right\}\,,$$

$$\mathcal{E}_n^+ := \left\{ f : f(t) = \sum_{j=1}^{n} a_j e^{\lambda_j t}\,, \quad a_j, \lambda_j \in \mathbb{C}\,,\ \ \mathrm{Re}(\lambda_j) \geq 0 \right\}\,,$$

and

$$\mathcal{T}_n := \left\{ f : f(t) = \sum_{j=1}^{n} a_j e^{i\lambda_j t}\,,\ \ a_j \in \mathbb{C},\ \ \lambda_1 < \lambda_2 < \cdots < \lambda_n \right\}\,.$$

A highlight of this paper is the asymptotically sharp inequality

$$|f(0)| \leq (1+\varepsilon_n)\, 3n\, \|f(t)e^{-9nt/2}\|_{L_2[0,1]}, \qquad f \in \mathcal{T}_n\,,$$

where ε_n converges to 0 rapidly as n tends to ∞. The inequality

$$\sup_{0 \not\equiv f \in \mathcal{T}_n} \frac{|f(0)|}{\|f\|_{L_2[0,1]}} \geq n$$

is also observed. Our results improve an old result of G. Halász and a recent result of G. Kós. We prove several other essentially sharp related results in this paper.

1. Introduction and Notation

The well known results of Nikolskii assert that the essentially sharp inequality

$$\|P\|_{L_q[-1,1]} \leq c(p,q) n^{2/p-2/q} \|P\|_{L_p[-1,1]}$$

Key words and phrases. exponential sums, Nikolskii, Bernstein, and Markov type inequalities, infinite-finite range inequalities.

2000 Mathematics Subject Classifications. 11C08, 41A17

holds for all algebraic polynomials P of degree at most n with complex coefficients and for all $0 < p \le q \le \infty$, while the essentially sharp inequality

$$\|Q\|_{L_q[-\pi,\pi]} \le c(p,q)n^{1/p-1/q}\|Q\|_{L_p[-\pi,\pi]}$$

holds for all trigonometric polynomials Q of degree at most n with complex coefficients and for all $0 < p \le q \le \infty$. The subject started with two remarkable papers, [25] and [29]. There are quite a few related papers in the literature, and several books discuss inequalities of this variety with elegant proofs; see [4] and [13], for example. In this paper we focus on the classes

$$\mathcal{E}_n := \left\{ f : f(t) = \sum_{j=1}^{n} a_j e^{\lambda_j t} \,, \quad a_j, \lambda_j \in \mathbb{C} \right\} ,$$

$$\mathcal{E}_n^+ := \left\{ f : f(t) = \sum_{j=1}^{n} a_j e^{\lambda_j t} \,, \quad a_j, \lambda_j \in \mathbb{C} \,, \;\; \mathrm{Re}(\lambda_j) \ge 0 \right\} ,$$

$$\mathcal{E}_n^- := \left\{ f : f(t) = \sum_{j=1}^{n} a_j e^{\lambda_j t} \,, \quad a_j, \lambda_j \in \mathbb{C} \,, \;\; \mathrm{Re}(\lambda_j) \le 0 \right\} ,$$

and

$$\mathcal{T}_n := \left\{ f : f(t) = \sum_{j=1}^{n} a_j e^{i\lambda_j t} \,, \;\; a_j \in \mathbb{C} \,, \;\; \lambda_1 < \lambda_2 < \cdots < \lambda_n \right\} .$$

These classes were studied in several publications; see [21], [23], [24], and [32], for example. For the sake of brevity let

$$\|f\|_A := \sup_{t \in A} |f(t)|$$

for a complex-valued function f defined on a set $A \subset \mathbb{R}$. Section 19.4 of Turán's book [32] refers to the following result of G. Halász:

$$|f(0)| \le cn^5 \|f\|_{[0,1]} \,, \qquad f \in \mathcal{E}_n^+ \,,$$

where $c > 0$ is an absolute constant. This was improved recently by G. Kós [21] to

$$|f(0)| \le 10 \, \frac{5n}{5n-1} n^2 \|f\|_{L_1[0,1]} \,, \qquad f \in \mathcal{E}_n^+ \,, \tag{1.1}$$

where cn^2 is the best possible size of the factor in this inequality. He also proved that

$$|f(0)| \le 2n \|f\|_{L_2[0,1]} \,, \qquad f \in \mathcal{E}_n^+ \,, \tag{1.2}$$

where cn is the best possible size of the factor in this inequality. The technique used in [21] is based on integrating discrete inequalities similar to Turán's first and second main theorems in the theory of power sums. This technique was also used by Tijdeman as it was demonstrated, for example in Section 27 of Turán's book [32]. This answers a question of S. Denisov asked from me in e-mail communications. I was not aware of the above results when I started to write this paper. In this paper we recapture the above inequalities with better constants for all $f \in \mathcal{T}_n$. Namely we prove that

$$|f(0)| \leq cn^2 \|f\|_{L_1[0,1]}\,, \qquad f \in \mathcal{T}_n\,, \tag{1.3}$$

with $c = 2 + \log 4 + \varepsilon_n = 3.3862\ldots$ and

$$|f(0)| \leq cn \|f\|_{L_2[0,1]}\,, \qquad f \in \mathcal{T}_n\,, \tag{1.4}$$

with $c = (2 + \log 4 + \varepsilon_n)^{1/2} = 1.8401\ldots$, where ε_n converges to 0 rapidly as n tends to ∞. S. Denisov [12] has just proved that the constant $c = (2 + \log 4 + \varepsilon_n)^{1/2} = 1.8401\ldots$ can be further improved to $c = \pi/2 = 1.5707\ldots$ in (1.4). Denisov's approach also uses a Halász-like construction first, which may be found in [20] and it also appears as Lemma 10.8 in [29], but after that it employs a duality argument and an old result of Lachance, Saff, and Varga [22], which is not used by Kós. We note that Denisov's improvement of (1.4) can also be seen for all $f \in \mathcal{E}_n^+$ by modifying Kós's approach. Indeed, it is proved in [22] that

$$\sigma_k := \min\left\{\|P(e^{it})\|_{[0,2\pi]} : P(0) = 1,\;\; P(1) = 0,\;\; P \in \mathcal{P}_k^c\right\} = \left(\sec \frac{\pi}{2(k+1)}\right)^{k+1}\,,$$

where $\mathcal{P}_k^c$ denotes the set of all algebraic polynomials of degree at most k with complex coefficients. Hence there are polynomials $H_k \in \mathcal{P}_k^c$ such that $H_k(0) = 1$ and

$$\|H_k(e^{it})\|_{[0,2\pi]} \leq \left(\sec \frac{\pi}{2(k+1)}\right)^{k+1} = \exp\left(\frac{\pi^2}{8k} + O\left(\frac{1}{k^2}\right)\right)\,.$$

Using the above $H_k \in \mathcal{P}_k^c$ instead of the $H_k \in \mathcal{P}_k^c$ in Kós's proof satisfying only

$$\|H_k(e^{it})\|_{[0,2\pi]} \leq \exp\left(\frac{2}{k}\right)\,,$$

we get Denisov's improvement of (1.4) can be extended to all $f \in \mathcal{E}_n^+$, that is,

$$|f(0)| \leq \frac{\pi n}{2}\, \|f\|_{L_2[0,1]}\,, \qquad f \in \mathcal{E}_n^+\,. \tag{1.5}$$

In Section 2.2 the infinite-finite range inequality

$$\int_0^\infty |f(t)|^2 e^{-t}\, dt \leq (1 + \varepsilon_n)^2 \int_0^{9n} |f(t)|^2 e^{-t}\, dt$$

is stated for every $f \in \mathcal{E}_n^-$, in particular, for every $f \in \mathcal{T}_n$, where $(1+\varepsilon_n)^2 := 1+8190\, e^{-n/10}$. As a consequence we prove that

$$|f(0)| \leq (1 + \varepsilon_n)\, 3n\, \|f(t)e^{-9nt/2}\|_{L_2[0,1]}\,, \qquad f \in \mathcal{T}_n\,,$$

where ε_n is the same as before, and for every $\lambda_1 < \lambda_2 < \cdots < \lambda_n$ there is an $f \in \mathcal{T}_n$ of the form

$$f(t) = \sum_{j=1}^{n} a_j e^{i\lambda_j t}\,, \qquad a_j \in \mathbb{C}\,, \tag{1.6}$$

such that

$$|f(0)| > 3n\, \|f(t)e^{-9nt/2}\|_{L_2[0,1]}\,.$$

Other Nikolskii-type inequalities comparing the $L_p[0,1]$ and $L_q[0,1]$ norms of exponential sums $f \in \mathcal{T}_n$ are also established in Section 2.1 We use quite different techniques based on the knowledge of Müntz-Legendre orthonormal polynomials studied in [9] and Section 3.4 of [4]. We obtain interesting Markov-type inequalities as well for the derivatives of exponential sums $f \in \mathcal{T}_n$, but such a Markov-type inequality cannot depend only on n, it depends on the exponents $\lambda_1 < \lambda_2 < \cdots < \lambda_n$. We also examine how far our estimates are from being sharp, and it turns out that our main results proved in this paper are essentially sharp. Most importantly, the inequality

$$\sup_{0 \not\equiv f \in \mathcal{T}_n} \frac{|f(0)|}{\|f\|_{L_2[0,1]}} \geq n\,.$$

is also observed in Section 2.1. The inequality

$$|f(0)| \leq n \|f\|_{L_2[0,1]}$$

for every $f \in \mathcal{E}_n^+$ of the form

$$f(t) = \sum_{j=1}^{n} a_j e^{\lambda_j t}\,, \qquad a_j \in \mathbb{R}\,,\ \ 0 \leq \lambda_1 < \lambda_2 < \cdots < \lambda_n\,, \tag{1.7}$$

is stated in Section 2.3. This inequality is sharp. We suspect that the above inequality holds for all $f \in \mathcal{T}_n$ or perhaps for all $f \in \mathcal{E}_n^+$ at least with n replaced by $(1+\varepsilon_n)n$, where ε_n tends to 0 as n tends to ∞. Markov-Nikolskii-type inequalities for $\mathcal{T}_n$ are established in Section 2.4. Markov-Nikolskii-type inequalities for $f \in \mathcal{E}_n$ with nonnegative exponents are formulated in Section 2.5. We claim that

$$|f'(0)| \leq (1+\varepsilon_n)\, 3^{-1/2}\, n^3 \|f\|_{L_2[0,1]}$$

for every $f \in \mathcal{E}_n^+$ of the form (1.7), where the quantity ε_n (determined exactly in the proof) tends to 0 an n tends to ∞. This inequality is sharp. Section 2.6 offers an essentially sharp pointwise Nikolskii-type inequality for $\mathcal{E}_n$, namely we claim that

$$\left(\frac{(n-2)\log 2}{4\min\{y-a, b-y\}}\right)^{1/2} \leq \sup_{0 \not\equiv f \in \mathcal{E}_n} \frac{|f(y)|}{\|f\|_{L_2[a,b]}} \leq \left(\frac{2n}{\min\{y-a, b-y\}}\right)^{1/2}$$

for every $y \in (a,b)$. In Section 2.7 we offer the Bernstein-type inequality

$$|f'(0)| \leq 2e(\lambda + n + 1)\, \|f\|_{[-1,1]}$$

for every $f \in \mathcal{T}_n$ of the form (1.6), where

$$\lambda := \max_{1 \leq j \leq n} |\lambda_j|\,, \qquad \lambda_1 < \lambda_2 < \cdots < \lambda_n\,. \tag{1.8}$$

This inequality is sharp up to the factor $2e$. Namely, for every real number $\lambda > 0$ and integer $n \geq 1$ there is an $f \in \mathcal{T}_n$ of the form (1.6) with (1.8) such that

$$|f'(0)| \geq \frac{1}{4}\,(\lambda + n - 3)\, \|f\|_{[-1,1]}\,.$$

In Section 2.8 the Markov-type inequality

$$\|f'\|_{[0,1]} \leq (1 + \varepsilon_n) \left(108n^5 + \sum_{k=1}^{n} \lambda_k^2 \right)^{1/2} \|f\|_{[0,1]}$$

for every $f \in \mathcal{T}_n$ of the form (1.6) is established, where the quantity ε_n (determined exactly in the proof) tends to 0 an n tends to ∞. We record an observation showing how far the above Markov inequality is from being sharp. Markov-type inequalities for $\mathcal{E}_n^-$ and $\mathcal{T}_n$ in $L_2[0,\infty)$ with the Laguerre weight are established in Section 2.9. Our Theorem 2.9.1 extends Lubinsky's Theorem 3.2 in [23] from the case of exponential sums with purely imaginary exponents to the case of exponential sums with complex exponents. Our only result in Section 2.10 is a version of Theorem 2.9.1, a Markov-type inequality for $\mathcal{E}_n^-$ in $L_2[0,\infty)$ without a weight. We prove our new results in Section 4. Lemmas needed in the proofs of our new results are stated and proved in Section 3. Combining Turán's power sum method with results in [10], [11], and [18], we may be able to prove other interesting results in the future. We close the paper with an Appendix listing results closely related to our new results in this paper. Theorems 5.1–5.6 have been proved by subtle Descartes system methods which can be employed only in the case of exponential sums with real exponents but not in the case of complex exponents. The reader may find it useful to compare the results in Section 5 with the new results of the paper.

Throughout the paper $\mathcal{P}_n^c$ denotes the set of all algebraic polynomials of degree at most n with complex coefficients, and $\mathcal{P}_n$ denotes the set of all algebraic polynomials of degree n with real coefficients. Observe that

$$t = \lim_{\varepsilon \to 0+} \frac{e^{i\varepsilon t} - 1}{i\varepsilon}$$

and the remark below follows immediately.

Remark 1.1. For every $P \in \mathcal{P}_{n-1}^c$ there are $f_k \in \mathcal{T}_n$ of the form

$$f_k(t) = P\left(\frac{e^{i\varepsilon_k t} - 1}{i\varepsilon_k} \right), \qquad \varepsilon_k > 0\,, \qquad \lim_{k \to \infty} \varepsilon_k = 0\,,$$

such that

$$\lim_{k \to \infty} \|(f_k(t) - P(t))e^{-t}\|_{[0,\infty)} = \lim_{k \to \infty} \|(f_k'(t) - P'(t))e^{-t}\|_{[0,\infty)} = 0\,.$$

2. New Results

2.1. Nikolskii-type inequalities for $\mathcal{T}_n$

In Section 5 we review certain Nikolskii-type and Markov-Nikolskii type inequalities known for exponential sums with only real exponents λ_j; see Theorems 5.1, 5.2, 5.3, and 5.4. What happens to Nikolskii-type inequalities if we consider exponential sums with purely imaginary, or more generally, arbitrary complex exponents? Answering a question of Sergey Denisov (e-mail communications) in this section first we establish a new Nikolskii-type inequality for exponential sums in $\mathcal{T}_n$. Observe that while our constant $(8+\varepsilon_n)$ in Theorem 2.1.1 is not as good as $\pi/2$ or even 2, there is a rapidly decreasing weight function $w(t) = e^{-nt}$ in Theorem 2.1.1 pushing the $L_2[0,1]$ norm down at the right-hand side.

Theorem 2.1.1. *We have*

$$|f(0)| \leq (8+\varepsilon_n)^{1/2}\, n\, \|f(t)e^{-nt}\|_{L_2[0,1]}, \qquad f \in \mathcal{T}_n\,,$$

where $(8+\varepsilon_n)^{1/2} := 8^{1/2}(1+2e^{-2n})^{1/2}$*, and for every* $\lambda_1 < \lambda_2 < \cdots < \lambda_n$ *there is an* $f \in \mathcal{T}_n$ *of the form (1.6) such that*

$$|f(0)| > 8^{1/2}\, n\, \|f(t)e^{-4nt}\|_{L_2[0,1]}\,.$$

Our next theorem recaptures Kós's inequality (1.1) with a constant better than $c = 2$ but not as good as $c = \pi/2$. The constant specified in our theorem below seems to be the limit of what our essentially different method based on the explicit form of Müntz-Legendre orthonormal polynomials gives.

Theorem 2.1.2. *Let* $\gamma_0 := 2 + \log 4 < \gamma \leq 4$*. We have*

$$|f(0)| \leq (\gamma+\varepsilon_n)^{1/2}\, n\, \|f\|_{L_2[0,1]}, \qquad f \in \mathcal{T}_n\,,$$

where

$$(\gamma+\varepsilon_n)^{1/2} = \gamma^{1/2}(1+\delta^{-2}e^{-\delta\gamma n})^{1/2}\,, \qquad \delta := \frac{\gamma-\gamma_0}{8}\,.$$

Observe that if $f \in \mathcal{T}_n$ and $g(t) = f(-t)$, then $g \in \mathcal{T}_n$. Hence the extension of Theorem 2.1.2 formulated by our next couple of theorems follows easily.

Theorem 2.1.3. *We have*

$$\|f\|_{[0,1]} \leq \frac{\pi n}{2}\, \|f\|_{L_2[0,1]}\,, \qquad f \in \mathcal{T}_n\,.$$

Theorem 2.1.4. *We have*

$$\|f\|_{[0,1]} \leq \left(\frac{\pi n}{2}\right)^{2/q} \|f\|_{L_q[0,1]}\,, \qquad f \in \mathcal{T}_n\,, \quad q \in (0,2]\,.$$

Theorem 2.1.5. *We have*

$$\|f\|_{L_p[0,1]} \leq \left(\frac{\pi n}{2}\right)^{2/q-2/p} \|f\|_{L_q[0,1]}\,, \qquad f \in \mathcal{T}_n\,, \quad 0 < q < p \leq \infty\,, \quad q \leq 2\,.$$

Note that the case $q = 1$ of Theorem 2.1.4 improves Kós's inequality (1.1) to

$$\|f\|_{[0,1]} \leq \frac{\pi^2 n^2}{4} \|f\|_{L_1[0,1]}\,, \qquad f \in \mathcal{T}_n\,.$$

Theorem 2.1.6. *We have*

$$\sup_{0 \not\equiv f \in \mathcal{T}_n} \frac{|f(0)|}{\|f\|_{L_2[0,1]}} \geq n\,.$$

Theorem 2.1.7. *There is an absolute constant $c > 0$ such that*

$$\sup_{0 \not\equiv f \in \mathcal{T}_n} \frac{|f(0)|}{\|f\|_{L_q[0,1]}} \geq c^{1+1/q}(1+qn)^{2/q}, \qquad q \in (0,\infty)\,.$$

Remark 2.1.8. It remains open what are the right extensions of Theorems 2.1.4 and 2.1.5 to $q > 2$. Note that Theorem 5.8 implies that

$$\sup_{0 \not\equiv f \in \mathcal{T}_n} \frac{|f(0)|}{\|f\|_{L_q[0,1]}} \geq c_q n^{1/2}, \qquad q \in (0,\infty)\,,$$

with a constant $c_q > 0$ depending only on $q > 0$. Hence

$$c_q n^{1/2} \leq \sup_{0 \not\equiv f \in \mathcal{T}_n} \frac{|f(0)|}{\|f\|_{L_q[0,1]}} \leq \sup_{0 \not\equiv f \in \mathcal{T}_n} \frac{|f(0)|}{\|f\|_{L_2[0,1]}} \leq \frac{\pi n}{2}\,.$$

In particular, Theorems 2.1.4 cannot remain true for $q > 4$. Nevertheless, we can prove the following two results.

Theorem 2.1.9. *We have*

$$|f(0)| \leq (8+\varepsilon_n)^{1/2}\, c_q n^{1/2+1/q} \|f(t)e^{-nt}\|_{L_q[0,1]}\,, \quad f \in \mathcal{T}_n\,, \quad q \in (2,\infty)\,,$$

where ε_n is the same as in Theorem 2.1.1 and

$$c_q := \left(\frac{q-2}{2q}\right)^{(q-2)/(2q)}\,.$$

Theorem 2.1.10. *We have*

$$\|f\|_{[0,1]} \leq (8+\varepsilon_n)^{1/2} c_q n^{1/2+1/q} \|f\|_{L_q[0,1]}\,, \quad f \in \mathcal{T}_n\,, \quad q \in (2,\infty)\,,$$

where ε_n is the same as in Theorem 2.1.1 and c_q is the same as in Theorem 2.1.9.

2.2. An infinite-finite range inequality for $\mathcal{E}_n^-$ with an application

Our next theorem is an infinite-finite range inequality for all $f \in \mathcal{E}_n^-$.

Theorem 2.2.1. *We have*

$$\int_0^\infty |f(t)|^2 e^{-t}\, dt \le (1+\varepsilon_n)^2 \int_0^{9n} |f(t)|^2 e^{-t}\, dt$$

for every $f \in \mathcal{E}_n^-$, in particular, for every $f \in \mathcal{T}_n$, where $(1+\varepsilon_n)^2 := 1 + 8190\, e^{-n/10}$.

The theorem below establishes an asymptotically sharp version of Kós's inequality $|f(0)| \le 2n\|f\|_{L_2[0,1]}$ in the presence of the rapidly decreasing weight function $w(t) = e^{-9nt/2}$ pushing the $L_2[0,1]$ norm down at the right-hand side.

Theorem 2.2.2. *Let ε_n be the same as in Theorem 2.2.1. We have*

$$|f(0)| \le (1+\varepsilon_n)\, 3n\, \|f(t)e^{-9nt/2}\|_{L_2[0,1]}\,, \qquad f \in \mathcal{T}_n\,,$$

and for every $\lambda_1 < \lambda_2 < \cdots < \lambda_n$ there is an $f \in \mathcal{T}_n$ of the form (1.6) such that

$$|f(0)| > 3n\, \|f(t)e^{-9nt/2}\|_{L_2[0,1]}\,.$$

2.3. A sharp Nikolskii-type inequality for $f \in \mathcal{E}_n$ with nonnegative exponents

Our next theorem establishes the best constant in the inequality $|f(0)| \le cn\|f\|_{L_2[0,1]}$ for functions f in a subclass of $\mathcal{E}_n$.

Theorem 2.3.1. *We have*

$$|f(0)| \le n\|f\|_{L_2[0,1]}$$

for every $f \in \mathcal{E}_n^+$ of the form

$$f(t) = \sum_{j=1}^n a_j e^{\lambda_j t}\,, \qquad a_j \in \mathbb{R}\,,\ \ 0 \le \lambda_1 < \lambda_2 < \cdots < \lambda_n\,.$$

This inequality is sharp.

2.4. Markov-Nikolskii-type inequalities for $\mathcal{T}_n$

The next theorem establishes an essentially sharp result when $|f(0)|$ is replaced by $|f'(0)|$ in Theorem 2.2.2.

Theorem 2.4.1. *Let ε_n be the same as in Theorem 2.2.1. We have*

$$|f'(0)| \leq 27\,(1+\varepsilon_n)\,n^{3/2} \left(\sum_{k=1}^{n} \left(\left(\frac{\lambda_k}{9n} \right)^2 + (k-1)^2 \right) \right)^{1/2} \|f(t)e^{-9nt/2}\|_{L_2[0,1]}$$

for every $f \in \mathcal{T}_n$ of the form (1.6), and for every $\lambda_1 < \lambda_2 < \cdots < \lambda_n$ there is an $f \in \mathcal{T}_n$ of the form (1.6) such that

$$|f'(0)| > 27\,n^{3/2} \left(\sum_{k=1}^{n} \left(\left(\frac{\lambda_k}{9n} \right)^2 + (k-1)^2 \right) \right)^{1/2} \|f(t)e^{-9nt/2}\|_{L_2[0,1]}\,.$$

The next theorem establishes an essentially sharp result when $|f'(0)|$ is replaced by $\|f'\|_{[0,1]}$ in Theorem 2.4.1.

Theorem 2.4.2. *Let ε_n be the same as in Theorem 2.2.1. We have*

$$\|f'\|_{[0,1]} \leq 27\,(1+\varepsilon_n)\,n^{3/2} \left(\sum_{k=1}^{n} \left(2\left(\frac{\lambda_k}{9n} \right)^2 + 8(k-1)^2 \right) \right)^{1/2} \|f\|_{L_2[0,1]}$$

for every $f \in \mathcal{T}_n$ of the form (1.6).

To formulate our next observation, given $n \in \mathbb{N}$ and $\eta > 0$, we introduce the classes

$$\mathcal{T}_n(\eta) := \left\{ f : f(t) = \sum_{j=1}^{n} a_j e^{i\lambda_j t}\,,\;\; a_j \in \mathbb{C}, \quad 0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n \leq \eta \right\}. \tag{2.4.1}$$

Theorem 2.4.3. *We have*

$$\sup_{0 \not\equiv f \in \mathcal{T}_n(\eta)} \frac{|f'(0)|}{\|f\|_{L_2[0,1]}} \geq (1+\varepsilon_n^*)\,3^{-1/2}\,n^3$$

for every $n \in \mathbb{N}$ and for every $\eta > 0$, where ε_n^* (determined exactly in the proof) is a quantity tending to 0 an n tends to ∞.

2.5. Markov-Nikolskii-type inequalities for $f \in \mathcal{E}_n$ with nonnegative exponents

Our next theorem records how large $|f'(0)|$ can be if $\|f\|_{L_2[0,1]} = 1$ for exponential sums $f \in \mathcal{E}_n$ with nonnegative exponents.

Theorem 2.5.1. *We have*

$$|f'(0)| \leq (1+\varepsilon_n^*)\,3^{-1/2}\,n^3 \|f\|_{L_2[0,1]}$$

for every $f \in \mathcal{E}_n^+$ of the form

$$f(t) = \sum_{j=1}^{n} a_j e^{\lambda_j t}\,, \qquad a_j \in \mathbb{R}\,,\;\; 0 \leq \lambda_1 < \lambda_2 < \cdots < \lambda_n\,,$$

where the quantity ε_n^* (determined exactly in the proof) tends to 0 an n tends to ∞. This inequality is sharp.

2.6. A pointwise Nikolskii-type inequality for $\mathcal{E}_n$

The upper bound of the theorem below follows from Lemma 3.5 proved in [7]. We couple this upper bound with a matching lower bound.

Theorem 2.6.1. *We have*

$$\left(\frac{(n-2)\log 2}{32\,\min\{y-a,b-y\}}\right)^{1/2} \leq \sup_{0\not\equiv f\in\mathcal{E}_n} \frac{|f(y)|}{\|f\|_{L_2[a,b]}} \leq \left(\frac{2n}{\min\{y-a,b-y\}}\right)^{1/2}$$

for every $y \in (a,b) \subset \mathbb{R}$.

The theorem below shows a lower bound for

$$\sup_{0\not\equiv f\in\mathcal{T}_n} \frac{|f(y)|}{\|f\|_{L_2[a,b]}}\,.$$

However, there is a gap between the lower bound of Theorem 2.6.2 and the upper bound of Theorem 2.6.1.

Theorem 2.6.2. *There is an absolute constant $c > 0$ such that*

$$c\,\min\left\{\frac{n^{1/2}}{(\min\{y-a,b-y\})^{1/4}}\,,\;\; \frac{n}{(b-a)^{1/2}}\right\} \leq \sup_{0\not\equiv f\in\mathcal{T}_n(\eta)} \frac{|f(y)|}{\|f\|_{L_2[a,b]}}$$

for every $\eta > 0$ and for every $y \in [a,b] \subset \mathbb{R}$, where the classes $\mathcal{T}_n(\eta)$ are defined by (2.4.1).

Note that for $0 < \eta_1 < \eta_2$ we have $\mathcal{T}_n(\eta_1) \subset \mathcal{T}_n(\eta_2)$, so the statement of Theorem 2.6.2 gets stronger as $\eta > 0$ gets smaller since the constant $c > 0$ in Theorem 2.6.2 is absolute (independent of n, a, b, y and η).

2.7. An essentially sharp Bernstein-type inequality for $\mathcal{T}_n$

Our next theorem may be viewed as an essentially sharp (up to the constant $2e$) Bernstein type inequality for all $f \in \mathcal{T}_n$ at least in the middle of the interval $[-1,1]$.

Theorem 2.7.1. *We have*

$$|f'(0)| \leq (\lambda + 2e(n+1))\,\|f\|_{[-1,1]}$$

for every $f \in \mathcal{T}_n$ of the form (1.6), where

$$\lambda := \max_{1\leq j\leq n} |\lambda_j|\,, \qquad \lambda_1 < \lambda_2 < \cdots < \lambda_n\,. \tag{2.7.1}$$

This inequality is sharp up to the factor $2e$. Namely, for every real number $\lambda > 0$ and integer $n \geq 1$ there is an $f \in \mathcal{T}_n$ of the form (1.6) with (2.7.1) such that

$$|f'(0)| \geq \frac{1}{4}\,(\lambda + n - 3)\,\|f\|_{[-1,1]}\,.$$

Theorem 2.7.1 should be compared with Theorem 5.6 establishing an essentially sharp Bernstein-type inequality for the classes

$$E_n := \left\{f : f(t) = a_0 + \sum_{j=1}^{n} a_j e^{\lambda_j t}\,,\;\; a_j, \lambda_j \in \mathbb{R}\right\}.$$

2.8. Markov-type inequality for $\mathcal{T}_n$

Our next theorem offers a Markov-type inequality for all $f \in \mathcal{T}_n$ on $[0,1]$.

Theorem 2.8.1. *Let ε_n be the same as in Theorems 2.2.1, 2.4.1, and 2.4.2. We have*

$$|f'(0)| \leq (1+\varepsilon_n)\left(27n^5 + \sum_{k=1}^{n} \lambda_k^2\right)^{1/2} \|f\|_{[0,1]},$$

and

$$\|f'\|_{[0,1]} \leq (1+\varepsilon_n)\left(108n^5 + \sum_{k=1}^{n} \lambda_k^2\right)^{1/2} \|f\|_{[0,1]},$$

for every $f \in \mathcal{T}_n$ of the form (1.6).

The theorem below shows how far Theorem 2.8.1 is from being sharp.

Theorem 2.8.2. *We have*

$$\sup_{0 \not\equiv f \in \mathcal{T}_n(\eta)} \frac{|f'(0)|}{\|f\|_{[0,1]}} \geq 2(n-1)^2$$

for every $n \in \mathbb{N}$ and for every $\eta > 0$, where the classes $\mathcal{T}_n(\eta)$ are defined by (2.4.1).

Note that for $0 < \eta_1 < \eta_2$ we have $\mathcal{T}_n(\eta_1) \subset \mathcal{T}_n(\eta_2)$, so the statement of Theorem 2.8.2 gets stronger as $\eta > 0$ gets smaller.

Theorem 2.8.1 should be compared with the $p = q = \infty$ case of Theorem 5.3 establishing an essentially sharp Markov-Nikolskii type inequality for the classes $E(\Lambda_n)$, where associated with a set of $\Lambda_n := \{\lambda_0, \lambda_1, \dots, \lambda_n\}$ of distinct real numbers

$$E(\Lambda_n) := \operatorname{span}\{e^{\lambda_0 t}, e^{\lambda_1 t}, \dots, e^{\lambda_n t}\} = \left\{ f : f(t) = \sum_{j=0}^{n} a_j e^{\lambda_j t},\ \ a_j \in \mathbb{R} \right\}. \tag{2.8.2}$$

2.9. Markov-type inequalities for $\mathcal{E}_n$ and $\mathcal{T}_n$ in $L_2[0,\infty)$ with the Laguerre weight

In this section we use the norm

$$\|f\|_2 := \left(\int_0^\infty |f(t)|^2 e^{-t}\, dt\right)^{1/2}.$$

Our first result extends Lubinsky's Theorem 3.2 in [23] to the case when the exponents are not necessarily purely imaginary.

Theorem 2.9.1. *We have*

$$\|f'\|_2 \leq \left(\max_{1\leq j\leq n} |\lambda_j| + \left(\sum_{j=1}^{n} (1-2\mathrm{Re}(\lambda_j)) \sum_{k=j+1}^{n} (1-2\mathrm{Re}(\lambda_k)) \right)^{1/2} \right) \|f\|_2$$

for every $f \in \mathcal{E}_n$ of the form

$$f(t) = \sum_{j=1}^{n} a_j e^{\lambda_j t}, \quad a_j, \lambda_j \in \mathbb{C}\,, \;\; \mathrm{Re}(\lambda_j) < 1/2\,.$$

The "empty sum" from $k = n+1$ to $k = n$ above is meant to be 0.

The theorem below recaptures Lubinsky's Theorem 3.2 in [23].

Theorem 2.9.2. *We have*

$$\|f'\|_2 \leq \left(\max_{1\leq j\leq n} |\lambda_j| + \left(\frac{n(n-1)}{2} \right)^{1/2} \right) \|f\|_2$$

for every $f \in \mathcal{T}_n$ of the form (1.6).

Our next result shows how far Theorems 2.9.2 and 2.9.1 are from being sharp.

Theorem 2.9.3. *We have*

$$\sup_{0\not\equiv f\in\mathcal{T}_n(\eta)} \frac{\|f'\|_2}{\|f\|_2} \geq \left(\eta^2 + \left(2\sin\frac{\pi}{4n-2} \right)^{-1} \right)^{1/2}$$

for every $\eta > 0$, where $\mathcal{T}_n(\eta)$ is defined by (2.4.1).

The proof of Theorem 2.9.3 depends heavily on a result of Turán. Improving a result of E. Schmidt, Turán [31] showed that

$$M_n := \sup_{0\not\equiv f\in\mathcal{P}_n} \frac{\|f'\|_2}{\|f\|_2} = \left(2\sin\frac{\pi}{4n+2} \right)^{-1}$$

and the extremal polynomial is

$$f(t) = \sum_{j=1}^{n} \left(\sin\frac{j\pi}{2n+1} \right) L_j(t)\,,$$

where $\mathcal{P}_n$ is the set of all algebraic polynomials of degree at most n with real coefficients and L_j is the j-th Laguerre polynomial.

We remark that the quantity

$$M_{n,k} := \sup_{0\not\equiv f\in\mathcal{P}_n} \frac{\max_{t\in[0,\infty)} |f'(t)e^{-t}|}{\max_{t\in[0,\infty)} |f(t)e^{-t}|}$$

was examined by Sklyarov [30], who proved that

$$\frac{8^k n! k!}{(n-k)!(2k)!} \left(1 - \frac{k}{2n} \right) \leq M_{n,k} \leq \frac{8^k n! k!}{(n-k)!(2k)!}$$

for all integers $n \geq 1$ and $k \geq 1$.

2.10. Markov-type inequalities for $\mathcal{E}_n^-$ in $L_2[0,\infty)$

Our only result in this section is a version of Theorem 2.9.1, a Markov-type inequality for $\mathcal{E}_n^-$ in $L_2[0,\infty)$ without a weight. Let

$$\|f\|_{L_2[0,\infty)} := \left(\int_0^\infty |f'(t)|^2\,dt\right)^{1/2}.$$

Theorem 2.10.1. *We have*

$$\|f'\|_{L_2[0,\infty)} \le \left(\frac{1}{2} + \max_{1\le j\le n}\left|\lambda_j + \frac{1}{2}\right| + 2\left(\sum_{j=1}^n \mathrm{Re}(\lambda_j)\sum_{k=j+1}^n \mathrm{Re}(\lambda_k)\right)^{1/2}\right)\|f\|_{L_2[0,\infty)}$$

for every $f \in \mathcal{E}_n^-$ of the form

$$f(t) = \sum_{j=1}^n a_j e^{\lambda_j t}, \quad a_j, \lambda_j \in \mathbb{C}\,, \ \ \mathrm{Re}(\lambda_j) < 0\,.$$

The "empty sum" from $k = n+1$ to $k = n$ above is meant to be 0.

3. Lemmas

Our first lemma is due to Turán; see E.6 b] on page 297 of [4]. In fact, this inequality plays a central role in Turán's book [32] as well.

Lemma 3.1. *We have*

$$|g(0)| \le \left(\frac{2e(\alpha+\beta)}{\beta}\right)^n \|g\|_{[\alpha,\alpha+\beta]}\,, \qquad g \in \mathcal{E}_n^+\,,$$

for every $\alpha > 0$ and $\beta > 0$.

In fact, we will need the following consequence of Lemma 3.1.

Lemma 3.2. *We have*

$$|f(t)| \le \left(\frac{2e(t-a)}{d}\right)^n \|f\|_{[a,a+d]} \le \left(\frac{2et}{d}\right)^n \|f\|_{[a,a+d]}\,, \qquad f \in \mathcal{E}_n^-\,,$$

for every $a > 0$, $d > 0$ and $t \ge a + d$.

Proof of Lemma 3.2. Let $f \in \mathcal{E}_n^-$. Let $g \in \mathcal{E}_n^+$ be defined by $g(x) := f(t-x)$. Associated with $a > 0$, $d > 0$, $t \ge a+d$ we define $\alpha := t-(a+d)$, $\beta := d$. Applying Lemma 3.1 with $g \in \mathcal{E}_n^+$ we get

$$\begin{aligned}|f(t)| = |g(0)| &\le \left(\frac{2e(\alpha+\beta)}{\beta}\right)^n \|g\|_{[\alpha,\alpha+\beta]} \\ &= \left(\frac{2e(t-a)}{\beta}\right)^n \|f\|_{[a,a+d]} \le \left(\frac{2et}{d}\right)^n \|f\|_{[a,a+d]}\end{aligned}$$

□

Our next lemma states the first inequality of part c] of E.2 coupled with part d] of E.2 on page 286 of [4]; see also Corollary 3.3 in [8].

Lemma 3.3. *We have*

$$\frac{|y^{1/2}P(y)|}{\|P\|_{L_2[0,1]}} \leq \left(\sum_{j=1}^{n}(1+2\mathrm{Re}(\lambda_j))\right)^{1/2}$$

for every Müntz polynomial $0 \not\equiv P$ *of the form*

$$P(x) = \sum_{j=1}^{n} a_j x^{\lambda_j}, \quad a_j, \lambda_j \in \mathbb{C}, \ \ \mathrm{Re}(\lambda_j) > -1/2\,,$$

and for every $y \in [0,1]$. *This inequality is sharp when* $y = 1$.

Using the substitution $x = e^{-t}$ Lemma 3.3 implies the following.

Lemma 3.4. *We have*

$$|f(0)| \leq \left(\sum_{j=1}^{n}(1-2\mathrm{Re}(\lambda_j))\right)^{1/2} \quad \|f(t)e^{-t/2}\|_{L_2[0,\infty)}$$

for every $f \in \mathcal{E}_n$ *of the form*

$$f(t) = \sum_{j=1}^{n} a_j e^{\lambda_j t} \quad a_j \in \mathbb{C}, \ \ \mathrm{Re}(\lambda_j) < 1/2\,.$$

This inequality is sharp.

The next lemma is from [7].

Lemma 3.5. *We have*

$$|f(y)| \leq \left(\frac{n}{\delta}\right)^{1/2} \|f\|_{L_2[y-\delta,y+\delta]}\,, \qquad f \in \mathcal{E}_n\,,$$

for every $y \in \mathbb{R}$ *and* $\delta > 0$.

Our next lemma states the second inequality of part c] of E.2 coupled with part d] of E.2 on page 286 of [4]; see also Corollary 3.3 in [9].

Lemma 3.6. *We have*

$$\frac{|y^{3/2}P'(y)|}{\|P\|_{L_2[0,1]}} \leq \left(\sum_{k=1}^{n}(1+2\mathrm{Re}(\lambda_k))\left|\lambda_k + \sum_{j=1}^{k-1}(1+2\mathrm{Re}(\lambda_j))\right|^2\right)^{1/2}$$

for every $y \in [0,1]$ *and for every Müntz polynomial* $0 \not\equiv P$ *of the form*

$$P(x) = \sum_{j=1}^{n} a_j x^{\lambda_j}, \quad a_j, \lambda_j \in \mathbb{C}, \ \ \mathrm{Re}(\lambda_j) > -1/2\,.$$

This inequality is sharp when $y = 1$.

Using the substitution $x = e^{-t}$ Lemma 3.6 implies the following.

Lemma 3.7. *We have*

$$\frac{|f'(0)|}{\|f(t)e^{-t/2}\|_{L_2[0,\infty)}} \leq \left(\sum_{k=1}^{n}(1+2\mathrm{Re}(\lambda_k))\left|\lambda_k+\sum_{j=1}^{k-1}(1+2\mathrm{Re}(\lambda_j))\right|^2\right)^{1/2}$$

for every exponential sums $0 \not\equiv f$ *of the form*

$$f(t)=\sum_{j=1}^{n} a_j e^{i\lambda_j t}, \quad a_j, \lambda_j \in \mathbb{C}, \ \ \mathrm{Re}(\lambda_j) < 1/2 .$$

This inequality is sharp.

Associated with a set $\Lambda_n := \{\lambda_0, \lambda_1, \ldots, \lambda_n\}$ of distinct real numbers let

$$E(\Lambda_n) := \mathrm{span}\{e^{\lambda_0 t}, e^{\lambda_1 t}, \ldots, e^{\lambda_n t}\} = \left\{ f : f(t) = \sum_{j=0}^{n} a_j e^{\lambda_j t}\,, \ \ a_j \in \mathbb{R} \right\} .$$

The heart of the proof of our Theorem 2.3.1 is the following pair of comparison lemmas. The proof of the next couple of lemmas is based on basic properties of Descartes systems, in particular on Descartes' Rule of Sign, and on a technique used earlier by P.W. Smith and Pinkus. Lorentz ascribes this result to Pinkus, although it was P.W. Smith [27] who published it. I have learned about the method of proofs of these lemmas from Peter Borwein, who also ascribes it to Pinkus. The proofs of these lemmas are stated as Lemmas 3.3 and 3.4 in [17], where their proofs are also presented. Section 3.2 of [4], for instance, gives an introduction to Descartes systems. Descartes' Rule of Signs is stated and proved on page 102 of [4].

Lemma 3.8. *Let* $\Delta_n := \{\delta_0 < \delta_1 < \cdots < \delta_n\}$ *and* $\Gamma_n := \{\gamma_0 < \gamma_1 < \cdots < \gamma_n\}$ *be sets of real numbers satisfying* $\delta_j \leq \gamma_j$ *for each* $j = 0, 1, \ldots, n$*. Let* $a, b, c \in \mathbb{R}$*,* $a < b \leq c$*. Let* w *be a not identically* 0*, continuous function defined on* $[a, b]$*. Let* $q \in (0, \infty]$*. We have*

$$\sup_{0 \not\equiv P \in E(\Delta_n)} \frac{|P(c)|}{\|Pw\|_{L_q[a,b]}} \leq \sup_{0 \not\equiv P \in E(\Gamma_n)} \frac{|P(c)|}{\|Pw\|_{L_q[a,b]}} .$$

Under the additional assumption $\delta_n \geq 0$ *we also have*

$$\sup_{0 \not\equiv P \in E(\Delta_n)} \frac{|P'(c)|}{\|Pw\|_{L_q[a,b]}} \leq \sup_{0 \not\equiv P \in E(\Gamma_n)} \frac{|P'(c)|}{\|Pw\|_{L_q[a,b]}} .$$

Lemma 3.9. *Let* $\Delta_n := \{\delta_0 < \delta_1 < \cdots < \delta_n\}$ *and* $\Gamma_n := \{\gamma_0 < \gamma_1 < \cdots < \gamma_n\}$ *be sets of real numbers satisfying* $\delta_j \leq \gamma_j$ *for each* $j = 0, 1, \ldots, n$*. Let* $a, b, c \in \mathbb{R}$*,* $c \leq a < b$*. Let* w *be a not identically* 0*, continuous function defined on* $[a, b]$*. Let* $q \in (0, \infty]$*. We have*

$$\sup_{0 \not\equiv P \in E(\Delta_n)} \frac{|P(c)|}{\|Pw\|_{L_q[a,b]}} \geq \sup_{0 \not\equiv P \in E(\Gamma_n)} \frac{|P(c)|}{\|Pw\|_{L_q[a,b]}} .$$

Under the additional assumption $\gamma_0 \leq 0$ we also have

$$\sup_{0 \not\equiv Q \in E(\Delta_n)} \frac{|Q'(c)|}{\|Qw\|_{L_q[a,b]}} \geq \sup_{0 \not\equiv Q \in E(\Gamma_n)} \frac{|Q'(c)|}{\|Qw\|_{L_q[a,b]}} \,.$$

An entire function f is said to be of exponential type τ if for any $\varepsilon > 0$ there exists a constant $k(\varepsilon)$ such that $|f(z)| \leq k(\varepsilon) e^{(\tau+\varepsilon)|z|}$ for all $z \in \mathbb{C}$. The following inequality may be found on p. 102 of [2] and is known as Bernstein's inequality; see also [3] and [14]. It can be viewed as an extension of Bernstein's (trigonometric) polynomial inequality (see p. 232 of [4], for instance) to entire functions of exponential type bounded on the real axis.

Lemma 3.10 (Bernstein's inequality). *We have*

$$\sup_{t \in \mathbb{R}} |f'(t)| \leq \tau \sup_{t \in \mathbb{R}} |f(t)| \,.$$

for every entire function f of exponential type $\tau > 0$ bounded on $\mathbb{R}$.

The reader may find another proof of the above Bernstein's inequality in [26, pp. 512–514], where it is also shown that an entire function f of exponential type τ satisfying

$$|f'(t_0)| = \tau \sup_{t \in \mathbb{R}} |f(t)|$$

at some point $t_0 \in \mathbb{R}$ is of the form

$$f(z) = ae^{i\tau z} + be^{-i\tau z} \,, \qquad a \in \mathbb{C}, \quad b \in \mathbb{C}, \qquad |a| + |b| = \sup_{t \in \mathbb{R}} |f(t)| \,.$$

Our next lemma is stated as Theorem 6.1.5 on page 282 of [4]; see also Theorem 3.4 in [9].

Lemma 3.11. *We have*

$$\frac{\|xP'(x)\|_{L_2[0,1]}}{\|P\|_{L_2[0,1]}} \leq \left(\sum_{j=1}^{n} |\lambda_j|^2 + \sum_{j=1}^{n} (1 + 2\mathrm{Re}(\lambda_j)) \sum_{k=j+1}^{n} (1 + 2\mathrm{Re}(\lambda_k)) \right)^{1/2}$$

for every Müntz polynomial $0 \not\equiv P$ of the form

$$P(x) = \sum_{j=1}^{n} a_j x^{\lambda_j} \,, \quad a_j, \lambda_j \in \mathbb{C} \,, \;\; \mathrm{Re}(\lambda_j) > -1/2 \,. \tag{3.1}$$

The "empty sum" from $k = n+1$ to $k = n$ above is meant to be 0.

In fact, a simple change in the proof (in either references) gives the following.

Lemma 3.12. *We have*

$$\frac{\|xP'(x)\|_{L_2[0,1]}}{\|P\|_{L_2[0,1]}} \leq \max_{1\leq j\leq n} |\lambda_j| + \left(\sum_{j=1}^{n}(1+2\mathrm{Re}(\lambda_j))\sum_{k=j+1}^{n}(1+2\mathrm{Re}(\lambda_k))\right)^{1/2}$$

for every Müntz polynomial $0 \not\equiv P$ *of the form (3.1). The "empty sum" from* $k = n+1$ *to* $k = n$ *above is meant to be* 0.

Proof of Lemma 3.12. Let P be a Müntz polynomial of the form (3.1). We have

$$P(x) = \sum_{k=1}^{n} a_k L_k^*, \qquad a_k \in \mathbb{C},$$

where

$$L_k^* \in \mathrm{span}\{x^{\lambda_1}, x^{\lambda_2}, \dots, x^{\lambda_k}\}$$

denotes the kth orthonormal Müntz-Legendre polynomials on $[0,1]$ associated with

$$\mathrm{span}\{x^{\lambda_1}, x^{\lambda_2}, \dots, x^{\lambda_n}\},$$

introduced in Section 3.4 of [4] (the spans here are taken over $\mathbb{C}$). Without loss of generality we may assume that

$$\|P\|_{L_2[0,1]} = \sum_{k=1}^{n} |a_k|^2 = 1\,. \tag{3.2}$$

As it is observed on page 283 of [4], we have

$$xP'(x) = \sum_{j=1}^{n}\left(a_j\lambda_j + \sqrt{1+2\mathrm{Re}(\lambda_j)}\sum_{k=j+1}^{n} a_k\sqrt{1+2\mathrm{Re}(\lambda_k)}\right)L_j^*(x)$$

Hence

$$\|xP'(x)\|_{L_2[0,1]} \leq \|R\|_{L_2[0,1]} + \|S\|_{L_2[0,1]}\,, \tag{3.3}$$

where

$$R(x) := \sum_{j=1}^{n} a_j\lambda_j L_j^*$$

and

$$S(x) := \sum_{j=1}^{n}\left(\sqrt{1+2\mathrm{Re}(\lambda_j)}\sum_{k=j+1}^{n} a_k\sqrt{1+2\mathrm{Re}(\lambda_k)}\right)L_j^*(x)\,.$$

Using the orthonormality of $\{L_j^*, j = 1, 2, \ldots, n\}$ on $[0, 1]$ and then recalling (3.2), we can deduce that

$$\|R\|_{L_2[0,1]} = \left(\sum_{j=1}^{n} |a_j \lambda_j|^2 \right)^{1/2} \leq \max_{1 \leq j \leq n} |\lambda_j| \left(\sum_{j=1}^{n} |a_j|^2 \right)^{1/2} \leq \max_{1 \leq j \leq n} |\lambda_j| \,. \tag{3.4}$$

Further, combining the orthonormality of $\{L_j^*, j = 1, 2, \ldots, n\}$ on $[0, 1]$ with applications of the Cauchy-Schwarz inequality to each term of the first sum and then recalling (3.2) we obtain that

$$\|S\|^2_{L_2[0,1]} = \sum_{j=1}^{n} (1 + 2\mathrm{Re}(\lambda_j)) \left| \sum_{k=j+1}^{n} a_k \sqrt{1 + 2\mathrm{Re}(\lambda_k)} \right|^2 \leq \sum_{j=1}^{n} (1 + 2\mathrm{Re}(\lambda_j)) \sum_{k=j+1}^{n} (1 + 2\mathrm{Re}(\lambda_k)) \tag{3.5}$$

The lemma now follows from (3.3), (3.4) and (3.5). $\square$

4. Proofs of the new results

Proof of Theorem 2.1.1. Let $f \in \mathcal{T}_n$. Applying Lemma 3.5 with $y \in [n, 7n]$ and $\delta := n$, we have

$$\|f\|_{[n,7n]} \leq \|f\|_{L_2[0,8n]} \,.$$

Combining this with Lemma 3.2 we get

$$|f(t)|^2 e^{-t} \leq \left(\left(\frac{2et}{6n} \right)^{2n} \|f\|^2_{[n,7n]} \right) e^{-t} \leq \left(\left(\frac{et}{3n} \right)^{2n} \|f\|^2_{L_2[0,8n]} \right) e^{-t} \leq e^{-t/2} \|f\|^2_{L_2[0,8n]} \,, \qquad t \geq 8n \,.$$

Here we used the fact that

$$h(t) := \left(\frac{et}{3n} \right)^{2n} e^{-t/2}$$

is decreasing on the interval $[8n, \infty)$, hence

$$\left(\frac{et}{3n} \right)^{2n} e^{-t} \leq \left(\left(\frac{et}{3n} \right)^{2n} e^{-t/2} \right) e^{-t/2} \leq \left(\frac{8e}{3} \right)^{2n} e^{-4n} e^{-t/2} \leq \left(\frac{(8/3)^2 e^2}{e^4} \right)^{n} \leq e^{-t/2} \,, \quad t \geq 8n \,.$$

Hence

$$\int_{8n}^{\infty} |f(t)|^2 e^{-t}\, dt \le \left(\int_{8n}^{\infty} e^{-t/2}\, dt\right) \|f\|^2_{L_2[0,8n]} \le 2e^{-2n} \int_0^{8n} |f(t)|^2 e^{-t/4}\, dt\,.$$

This implies that

$$\int_0^{\infty} |f(t)|^2 e^{-t}\, dt \le (1 + 2e^{-2n}) \int_0^{8n} |f(t)|^2 e^{-t/4}\, dt\,.$$

Combining this with Lemma 3.4 we get

$$|f(0)| \le (1 + 2e^{-2n})^{1/2}\, n^{1/2}\, \|f(t)e^{-t/8}\|_{L_2[0,8n]}$$

Transforming this inequality linearly from the interval $[0, 8n]$ to the interval $[0, 1]$, we get the first statement of the theorem.

The second statement of the theorem follows from the second statement of Lemma 3.4. Indeed, for every fixed $\lambda_1 < \lambda_2 < \cdots < \lambda_n$ there is a $0 \not\equiv f \in \mathcal{T}_n$ of the form (1.6) such that

$$|f(0)| \ge n^{1/2}\, \|f(t)e^{-t/2}\|_{L_2[0,\infty)} > n^{1/2}\, \|f(t)e^{-t/2}\|_{L_2[0,8n]}\,.$$

Transforming this inequality linearly from the interval $[0, 8n]$ to the interval $[0, 1]$, we get the second statement of the theorem. $\square$

Proof of Theorem 2.1.2. Let $\gamma_0 := 2 + \log 4 < \gamma \le 4$ and $\delta := (\gamma - \gamma_0)/8 < 1/8$. Observe that $\gamma_0 < \gamma \le 4$ implies that $0 < \delta < 1/8$ and hence

$$\gamma - 2\delta \ge \gamma_0 - 2\delta\gamma_0 - 1/4 > 2\,.$$

Combining this with the Mean Value Theorem we obtain

$$\log \gamma - \log(\gamma - 2\delta) < 2\delta\, \frac{1}{\gamma - 2\delta} < 2\delta\, \frac{1}{2} = \delta\,.$$

Therefore

$$\begin{aligned} 2 + \log 4 + 2 \log \frac{\gamma}{\gamma - 2\delta} - \gamma + \gamma\delta &= (\gamma_0 - \gamma) + 2(\log \gamma - \log(\gamma - 2\delta)) + \gamma\delta \\ &< -8\delta + 2\delta + 4\delta = -2\delta < 0\,, \end{aligned}$$

hence

$$4e^2 \left(\frac{\gamma}{\gamma - 2\delta}\right)^2 e^{\gamma(\delta - 1)} \le 1\,. \tag{4.1}$$

Let $f \in \mathcal{T}_n$. By Lemma 3.5 we have

$$\|f\|^2_{[\delta n, (\gamma - \delta)n]} \le \delta^{-1} \|f\|^2_{L_2[0,\gamma n]}\,.$$

Combining this with Lemma 3.2 we get

$$\begin{aligned}
|f(t)|^2 e^{-t} &\le \left(\left(\frac{2e(t-\delta n)}{(\gamma - 2\delta)n} \right)^{2n} \|f\|^2_{[\delta n, (\gamma-\delta)n]} \right) e^{-t} \\
&\le \delta^{-1} \left(\left(\frac{2et}{(\gamma - 2\delta)n} \right)^{2n} \|f\|^2_{L_2[0,\gamma n]} \right) e^{-t} \\
&\le \delta^{-1} e^{-\delta t} \|f\|^2_{L_2[0,\gamma n]} \,, \qquad t \ge \gamma n \,.
\end{aligned}$$

Here we used the fact that

$$h(t) := \left(\frac{2et}{(\gamma - 2\delta)n} \right)^{2n} e^{(\delta-1)t}$$

is decreasing on the interval $[\gamma, \infty) \subset [2(1-\delta)^{-1}, \infty)$, which, together with (4.1) yields

$$\begin{aligned}
\left(\frac{2et}{(\gamma - 2\delta)n} \right)^{2n} e^{-t} &\le \left(\left(\frac{2e\gamma}{\gamma - 2\delta)} \right)^{2n} e^{(\delta-1)t} \right) e^{-\delta t} \left(\left(\frac{2e\gamma}{\gamma - 2\delta)} \right)^{2n} e^{\gamma(\delta-1)n} \right) e^{-\delta t} \\
&\le \left(4e^2 \left(\frac{\gamma}{\gamma - 2\delta} \right)^2 e^{\gamma(\delta-1)} \right)^n e^{-\delta t} \le e^{-\delta t} \quad t \ge \gamma n \,,
\end{aligned}$$

Hence

$$\begin{aligned}
\int_{\gamma n}^{\infty} |f(t)|^2 e^{-t} \, dt &\le \delta^{-1} \left(\int_{\gamma n}^{\infty} e^{-\delta t} \, dt \right) \|f\|^2_{L_2[0,\gamma n]} \\
&\le \delta^{-1} \delta^{-1} e^{-\delta \gamma n} \int_0^{\gamma n} |f(t)|^2 \, dt \,.
\end{aligned}$$

This implies that

$$\int_0^{\infty} |f(t)|^2 e^{-t} \, dt \le (1 + \delta^{-2} e^{-\delta \gamma n}) \int_0^{\gamma n} |f(t)|^2 \, dt \,.$$

Combining this with Lemma 3.4 we get

$$|f(0)| \le n^{1/2} \, \|f(t) e^{-t/2}\|_{L_2[0,\infty)} \le n^{1/2} (1 + \delta^{-2} e^{-\delta \gamma n})^{1/2} \|f\|_{L_2[0,\gamma n]} \,.$$

Transforming this inequality linearly from the interval $[0, \gamma n]$ to the interval $[0, 1]$, we get the theorem. $\square$

Proof of Theorem 2.1.3. Let $y \in [-1, 1]$. Transforming the inequality of Theorem 2.1.1 (with the constant $\pi/2$ rather than $(\gamma + \varepsilon_n)^{1/2}$) linearly to the intervals $[0, y]$ and $[y, 1]$, respectively, we get

$$y \, |f(y)|^2 \le \left(\frac{\pi n}{2} \right)^2 \int_{[0,y]} |f(t)|^2 \, dt$$

and

$$(1-y)\,|f(y)|^2 \le \left(\frac{\pi n}{2}\right)^2 \int_{[y,1]} |f(t)|^2\,dt\,.$$

Adding these, we conclude that

$$|f(y)|^2 \le \left(\frac{\pi n}{2}\right)^2 \int_{[0,1]} |f(t)|^2\,dt\,,$$

and the theorem follows. □

Proof of Theorem 2.1.4. Let $f \in \mathcal{T}_n$ and $q \in (0,2]$. Using Theorem 2.1.3 we obtain

$$\begin{aligned}\|f\|_{[0,1]} \le & \frac{\pi n}{2}\,\|f\|_{L_2[0,1]} = \frac{\pi n}{2}\,\left(\int_0^1 |f(t)|^2\,dt\right)^{1/2} \\ & \le \frac{\pi n}{2}\,\left(\int_0^1 |f(t)|^q \|f\|_{[0,1]}^{2-q}\,dt\right)^{1/2}\,,\end{aligned}$$

and hence

$$\|f\|_{[0,1]}^{q/2} \le \frac{\pi n}{2}\,\|f\|_{L_q[0,1]}^{q/2}\,,$$

and the theorem follows. □

Proof of Theorem 2.1.5. When $p = \infty$ and $q \in (0,2]$, the theorem follows from Theorem 2.1.4. Let $0 < q < p < \infty$, $q \le 2$, and $f \in \mathcal{T}_n$. Based on Theorem 2.1.4 the proof of the theorem is fairly routine. We have

$$\begin{aligned}\|f\|_{L_p[0,1]}^p = & \int_{[0,1]} |f(t)|^p\,dt \le \int_{[0,1]} |f(t)|^q \|f\|_{[0,1]}^{p-q}\,dt \\ & \le \|f\|_{L_q[0,1]}^q \|f\|_{[0,1]}^{p-q} \le \|f\|_{L_q[0,1]}^q \left(\frac{\pi n}{2}\right)^{(p-q)2/q} \|f\|_{L_q[0,1]}^{p-q} \\ & \le \left(\frac{\pi n}{2}\right)^{(p-q)2/q} \|f\|_{L_q[0,1]}^p\,,\end{aligned}$$

and by taking the pth root of both sides the theorem follows. □

Proof of Theorem 2.1.6. The remark following Theorem 7.17.1 on page 182 of [28] asserts that

$$\sup_{0 \not\equiv P \in \mathcal{P}_n} \frac{|P(1)|}{\|P\|_{L_2[-1,1]}} = \sup_{0 \not\equiv P \in \mathcal{P}_n} \frac{\|P\|_{[-1,1]}}{\|P\|_{L_2[-1,1]}} = 2^{-1/2}(n+1)\,.$$

Transformation this linearly from $[-1,1]$ to $[0,1]$, we get

$$\sup_{0 \not\equiv P \in \mathcal{P}_{n-1}} \frac{|P(0)|}{\|P\|_{L_2[0,1]}} = \sup_{0 \not\equiv P \in \mathcal{P}_{n-1}} \frac{\|P\|_{[0,1]}}{\|P\|_{L_2[0,1]}} = n\,.$$

Now the theorem follows from Remark 1.1. □

Proof of Theorem 2.1.7. It is well-known; see Theorem 2.1 in [19] or the guided exercise E.19 on page 413 of [4], for instance, that there is an absolute constant $c > 0$ such that

$$\sup_{0 \not\equiv P \in \mathcal{P}_n} \frac{|P(1)|}{\|P\|_{L_q[-1,1]}} = \sup_{0 \not\equiv P \in \mathcal{P}_n} \frac{\|P\|_{[-1,1]}}{\|P\|_{L_q[-1,1]}} \geq c^{1+1/q}(1+qn)^{2/q}$$

for every $q \in (0,\infty)$. Transforming this linearly from $[-1,1]$ to $[0,1]$ we get that there is an absolute constant $c > 0$ such that

$$\sup_{0 \not\equiv P \in \mathcal{P}_{n-1}} \frac{|P(0)|}{\|P\|_{L_q[0,1]}} = \sup_{0 \not\equiv P \in \mathcal{P}_{n-1}} \frac{\|P\|_{[0,1]}}{\|P\|_{L_q[0,1]}} \geq c^{1+1/q}(1+qn)^{2/q}$$

for every $q \in (0,\infty)$. Now the theorem follows from Remark 1.1. □

Proof of Theorem 2.1.9. Let $q \in (2,\infty)$ and let $1/p := (q-2)/q$, that is, $1/p + 1/(q/2) = 1$. Using Theorem 2.1.1 and Hölder's inequality, we have

$$\begin{aligned}
|f(0)|^2 &\leq (8+\varepsilon_n)\, n^2 \int_0^1 |f(t)|^2 e^{-nt} e^{-nt}\, dt \\
&\leq (8+\varepsilon_n)\, n^2 \left(\int_0^1 \left(|f(t)|^2 e^{-nt}\right)^{q/2} dt \right)^{2/q} \left(\int_0^1 \left|e^{-nt}\right|^p dt \right)^{1/p},
\end{aligned}$$

hence

$$\begin{aligned}
|f(0)| &\leq (8+\varepsilon_n)^{1/2}\, n \|f(t)e^{-nt}\|_{L_q[0,1]} \left(\frac{1}{pn}\right)^{1/p} \\
&\leq (8+\varepsilon_n)^{1/2}\, n\, \|f(t)e^{-nt}\|_{L_q[0,1]} \left(\frac{q-2}{2qn}\right)^{(q-2)/q} \\
&\leq (8+\varepsilon_n)^{1/2}\, c_q n^{1/2+1/q} \|f(t)e^{-nt}\|_{L_q[0,1]}\,.
\end{aligned}$$

□

Proof of Theorem 2.1.10. Let $y \in [0,1]$. Transforming the inequality of Theorem 2.1.9 linearly to the intervals $[0,y]$ and $[y,1]$, respectively, we obtain that

$$y\, |f(y)|^q \leq \left((8+\varepsilon_n)^{1/2} c_q n^{1/2+1/q} \right)^q \int_0^y |f(t)|^q\, dt$$

and

$$(1-y)\, |f(y)|^q \leq \left((8+\varepsilon_n)^{1/2} c_q n^{1/2+1/q} \right)^q \int_y^1 |f(t)|^q\, dt\,.$$

Adding these we conclude that

$$|f(y)|^q \leq \left((8+\varepsilon_n)^{1/2} c_q n^{1/2+1/q} \right)^q \int_0^1 |f(t)|^q\, dt\,,$$

and the theorem follows. $\square$

Proof of Theorem 2.2.1. Let $f \in \mathcal{E}_n^-$. Let $\delta := 1/91$ and $\eta := 1/90$. By Lemma 3.5 we have

$$\|f\|_{[\delta n,(2-\delta)n]} \leq \delta^{-1}\|f\|_{L_2[0,2n]}\,.$$

Combining this with Lemma 3.2 we get

$$\begin{aligned}|f(t)|^2 e^{-t} &\leq \left(\left(\frac{2et}{(2-2\delta)n}\right)^{2n}\|f\|^2_{[\delta n,(2-\delta)n]}\right)e^{-t} \leq \left(\left(\frac{2et}{(2-2\delta)n}\right)^{2n}\delta^{-1}\|f\|^2_{L_2[0,2n]}\right)e^{-t}\\ &\leq \delta^{-1}\left(\frac{2et}{(2-2\delta)n}\right)^{2n} e^{2n}e^{-t}\left(\int_0^{2n}|f(x)|^2e^{-x}\,dx\right), \qquad t \geq 2n\,.\end{aligned}$$

Integrating on $[9n,\infty]$, we get

(4.2)

$$\begin{aligned}\int_{9n}^{\infty}|f(t)|^2e^{-t}\,dt \leq\ & \delta^{-1}\left(\int_{9n}^{\infty}\left(\frac{2et}{(2-2\delta)n}\right)^{2n}e^{2n}e^{-t}\,dt\right)\left(\int_0^{2n}|f(x)|^2e^{-x}\,dx\right)\\ =&\,\delta^{-1}\left(\sup_{t\geq 9n}\left(\frac{2et}{(2-2\delta)n}\right)^{2n}e^{2n}e^{(\eta-1)t}\,dt\right)\left(\int_{9n}^{\infty}e^{-\eta t}\,dt\right)\left(\int_0^{2n}|f(x)|^2e^{-x}\,dx\right)\\ \leq&\,\delta^{-1}\left(\int_{9n}^{\infty}e^{-\eta t}\,dt\right)\left(\int_0^{2n}|f(x)|^2e^{-x}\,dx\right)\\ \leq&\,\delta^{-1}\eta^{-1}e^{-9\eta n}\left(\int_0^{2n}|f(x)|^2e^{-x}\,dx\right).\end{aligned}$$

Here we used the fact that

$$h(t) := \left(\frac{2et}{(2-2\delta)n}\right)^{2n}e^{2n}e^{(\eta-1)t}$$

is decreasing on the interval $[9n,\infty)$, hence recalling that $\delta := 1/91$ and $\eta = 1/90$, we have

$$\sup_{t\geq 9n} h(t) \leq ((9.1)e)^{2n}e^{-(8.9)n}e^{2n} = e^{(2\log(9.1)+2-8.9+2)n} \leq e^0 = 1\,.$$

It follows from (4.3) that

$$\int_{9n}^{\infty}|f(t)|^2e^{-t}\,dt \leq \delta^{-1}\eta^{-1}e^{-9\eta n}\left(\int_0^{2n}|f(x)|^2e^{-x}\,dx\right),$$

hence

$$\int_0^{\infty}|f(t)|^2e^{-t}\,dt \leq (1+\delta^{-1}\eta^{-1}e^{-9\eta n})\left(\int_0^{9n}|f(x)|^2e^{-x}\,dx\right).$$

□

Proof of Theorem 2.2.2. Let $f \in \mathcal{T}_n$. Lemma 3.4 yields that

$$|f(0)|^2 \leq n \int_0^\infty |f(t)|^2 e^{-t}\, dt\,.$$

Combining this with Theorem 2.2.1 we have

$$|f(0)|^2 \leq (1+\varepsilon_n)^2 n \int_0^{9n} |f(t)|^2 e^{-t}\, dt\,.$$

Transforming this inequality from the interval $[0, 9n]$ to the interval $[0, 1]$, we obtain

$$|f(0)|^2 \leq (1+\varepsilon_n)^2 9n^2 \int_0^1 |f(u)|^2\, e^{-9nu}\, du\,.$$

The second statement of the theorem follows from the second statement of Lemma 3.4. Indeed, for every fixed $\lambda_1 < \lambda_2 < \cdots < \lambda_n$ there is a $0 \not\equiv f \in \mathcal{T}_n$ of the form (1.6) such that

$$|f(0)| = n^{1/2}\, \|f(t)e^{-t/2}\|_{L_2[0,\infty)} > n^{1/2}\, \|f(t)e^{-t/2}\|_{L_2[0,9n]}\,.$$

Transforming this inequality linearly from the interval $[0, 9n]$ to the interval $[0, 1]$, we get the second statement of the theorem. □

Proof of Theorem 2.3.1. Observe that

$$t = \lim_{\varepsilon \to 0+} \frac{e^{\varepsilon t} - 1}{\varepsilon}\,,$$

Hence it follows from Lemma 3.8 in a routine fashion that it is sufficient to prove the inequality only for polynomials $P \in \mathcal{P}_{n-1}$, where $\mathcal{P}_{n-1}$ denotes the set of all polynomials of degree at most $n-1$ with real coefficients, and this has been done in the proof of Theorem 2.1.6. The sharpness of the theorem also follows from the proof of Theorem 2.1.6. □

Proof of Theorem 2.4.1. Let $f \in \mathcal{T}_n$ be of the form (1.6), and let $g \in \mathcal{T}_n$ be defined by $g(9nt) := f(t)$. By Theorem 2.2.1 we have

$$\int_0^\infty |g(t)|^2 e^{-t}\, dt \leq (1+\varepsilon_n)^2 \int_0^{9n} |g(t)|^2 e^{-t}\, dt\,.$$

Combining this with Lemma 3.7 we get

$$\begin{aligned}
|f'(0)| &= 9n|g'(0)| \\
&\leq 9n \left(\sum_{k=1}^n \left(\left(\frac{\lambda_k}{9n}\right)^2 + (k-1)^2 \right) \right)^{1/2} \|g(t)e^{-t/2}\|_{L_2[0,\infty)} \\
&\leq 9n(1+\varepsilon_n) \left(\sum_{k=1}^n \left(\left(\frac{\lambda_k}{9n}\right)^2 + (k-1)^2 \right) \right)^{1/2} \|g(t)e^{-t/2}\|_{L_2[0,9n]} \\
&= 9n(1+\varepsilon_n) \left(\sum_{k=1}^n \left(\left(\frac{\lambda_k}{9n}\right)^2 + (k-1)^2 \right) \right)^{1/2} 3n^{1/2}\|f(u)e^{-9nu/2}\|_{L_2[0,1]}\,.
\end{aligned}$$

The second statement of the theorem follows from the second statement of Lemma 3.7. Indeed, for every fixed $\lambda_1 < \lambda_2 < \cdots < \lambda_n$ there is a $g \in \mathcal{T}_n$ such that $f \in \mathcal{T}_n$ defined by $g(9nt) := f(t)$ is of the form (1.6) and

$$\begin{aligned}
|f'(0)| &= 9n|g'(0)| \\
&= 9n \left(\sum_{k=1}^{n} \left(\left(\frac{\lambda_k}{9n} \right)^2 + (k-1)^2 \right) \right)^{1/2} \|g(t)e^{-t/2}\|_{L_2[0,\infty)} \\
&> 9n \left(\sum_{k=1}^{n} \left(\left(\frac{\lambda_k}{9n} \right)^2 + (k-1)^2 \right) \right)^{1/2} 3n^{1/2}\|f(u)e^{-9nu/2}\|_{L_2[0,1]} \, .
\end{aligned}$$

□

Proof of Theorem 2.4.2. Let $y \in [0,1]$. Transforming the inequality of Theorem 2.4.1 linearly to the intervals $[0,y]$ and $[y,1]$, respectively, we obtain that

$$y^3 \, |f'(y)|^2 \leq 27^2(1+\varepsilon_n)^2 \, n^3 \left(\sum_{k=1}^{n} \left(\left(\frac{y\lambda_k}{9n} \right)^2 + (k-1)^2 \right) \right)^{1/2} \int_0^y |f(t)|^2 \, dt$$

and

$$(1-y)^3 \, |f'(y)|^2 \leq 27^2(1+\varepsilon_n)^2 \, n^3 \left(\sum_{k=1}^{n} \left(\left(\frac{(1-y)\lambda_k}{9n} \right)^2 + (k-1)^2 \right) \right)^{1/2} \int_y^1 |f(t)|^2 \, dt \, .$$

Using the first inequality above if $y \in [1/2, 1]$ and the second inequality above if $y \in [1/2, 1]$ we conclude that

$$|f'(y)|^2 \leq 27^2 \, (1+\varepsilon_n)^2 \, n^3 \left(\sum_{k=1}^{n} \left(2 \left(\frac{\lambda_k}{9n} \right)^2 + 8(k-1)^2 \right) \right) \int_0^1 |f(t)|^2 \, dt \, ,$$

and the theorem follows. □

Proof of Theorem 2.4.3. By Remark 1.1 we have

$$\sup_{0 \not\equiv f \in \mathcal{T}_n} \frac{|f'(0)|}{\|f\|_{L_2[0,1]}} = \sup_{0 \not\equiv P \in \mathcal{P}_{n-1}^c} \frac{|P'(0)|}{\|P\|_{L_2[0,1]}} \, . \tag{4.3}$$

Let $P_n \in \mathcal{P}_n$ be the n-th orthonormal Legendre polynomial on the interval $[0,1]$, that is,

$$\int_0^1 P_n(x) P_m(x) \, dx = \delta_{n,m} \, .$$

where $\delta_{n,m} = 1$ if $n = m$ and $\delta_{n,m} = 0$ if $n \neq m$. Recall that

$$P_k'(0) = (-1)^k k(k+1)(2k+1)^{1/2} \, , \qquad k = 0, 1, \ldots \, . \tag{4.4}$$

This can be seen by combining (4.21.7), (4.3.3), and (4.1.4) in [27] and by using a linear transformation from the interval $[-1,1]$ to the interval $[0,1]$. As a consequence of orthonormality, the Cauchy-Schwarz inequality, and (4.4) it is well known (see E.2 on page 285 of [4], for instance) that

(4.5)
$$\sup_{0\not\equiv P\in\mathcal{P}_{n-1}} \frac{|P'(0)|}{\|P\|_{L_2[0,1]}} = \left(\sum_{k=0}^{n-1} P_k'(1)^2\right)^{1/2} = \left(\sum_{k=0}^{n-1} k^2(k+1)^2(2k+1)\right)^{1/2}$$
$$= (1+\varepsilon_n^*)\, 3^{-1/2}\, n^3\,.$$

Combining (4.3) and (4.5) gives the theorem. $\square$

Proof of Theorem 2.5.1. It follows from Lemma 3.9 in that it is sufficient to prove the inequality of the theorem for exponential sums $f\in E(\Delta_{n-1})$, where

$$\Delta_{n-1} := \Delta_{n-1}^{\epsilon} := \{\delta_0 < \delta_1 < \cdots < \delta_{n-1}\}$$

with
$$\delta_j := j\varepsilon\,, \qquad j = 0,1,\dots,n-1$$

and $\varepsilon > 0$ is sufficiently small. Observe that

$$t = \lim_{\varepsilon\to 0} \frac{e^{\varepsilon t}-1}{\varepsilon}\,,$$

hence for every $P\in\mathcal{P}_{n-1}$ there are $f_k\in E(\Delta_{n-1}^{\varepsilon_k})$ of the form

$$f_k(t) = P\left(\frac{e^{\varepsilon_k t}-1}{\varepsilon_k}\right)\,, \qquad \varepsilon_k > 0$$

such that
$$\lim_{k\to\infty} \|f_k - P\|_{[0,1]} = \lim_{k\to\infty} \|f_k' - P'\|_{[0,1]} = 0\,.$$

Hence, it is sufficient to prove the inequality of the theorem only for polynomials $P\in\mathcal{P}_{n-1}$. Therefore (4.5) gives the theorem. $\square$

Proof of Theorem 2.6.1. The upper bound follows from Lemma 3.5; see [7] for a proof. To see the lower bound we proceed as follows. Let $P_n\in\mathcal{P}_n$ be the n-th orthonormal Legendre polynomial on the interval $[-1,1]$, that is,

$$\int_{-1}^{1} P_n(x)P_m(x)\,dx = \delta_{n,m}\,,$$

where $\delta_{n,m} = 1$ if $n = m$ and $\delta_{n,m} = 0$ if $n \neq m$. Let

(4.6)
$$Q(x) = \sum_{k=0}^{n} P_k(0)P_k(x)$$

We have

$$(4.7) \qquad \|Q\|^2_{L_2[-1,1]} = \sum_{k=0}^{n} P_k(0)^2 \qquad \text{and} \qquad |Q(0)| = \sum_{k=0}^{n} P_k(0)^2\,,$$

hence

$$\frac{|Q(0)|^2}{\|Q\|^2_{L_2[-1,1]}} = \sum_{k=0}^{n} L_k(0)^2\,.$$

It is well known (see p. 165 of [28], for example) that $P_k(0) = 0$ if k is even, and

$$\begin{aligned}
|P_k(0)|^2 &= \frac{2k+1}{2}\left(\frac{1}{2}\right)^2\left(\frac{3}{4}\right)^2\left(\frac{5}{6}\right)^2\cdots\left(\frac{k-3}{k-2}\right)^2\left(\frac{k-1}{k}\right)^2 \\
&\geq \left(\frac{1}{2}\right)^2\left(\frac{2}{3}\,\frac{3}{4}\right)\left(\frac{4}{5}\,\frac{5}{6}\right)\cdots\left(\frac{k-4}{k-3}\,\frac{k-3}{k-2}\right)\left(\frac{k-2}{k-1}\,\frac{k-1}{k}\right) \\
&\geq \frac{2k+1}{4k} \geq \frac{1}{2}
\end{aligned}$$

if k is odd. Combining this with (4.6) and (4.7) gives

$$\frac{|Q(0)|^2}{\|Q\|^2_{L_2[-1,1]}} \geq \frac{n-2}{4}\,.$$

Let $f(t) = Q(2e^{-t}-1)e^{-t/2}$. We have

$$\frac{|f(\log 2)|}{\|f\|_{L_2[0,\infty)}} = \frac{|Q(0)|}{2^{1/2}\,\|Q\|_{L_2[-1,1]}} \geq \frac{(n-2)^{1/2}}{8^{1/2}}\,.$$

Transforming the above inequality linearly from the interval $[0,\infty)$ to $[a,\infty)$ and $(-\infty,b]$, we get the the lower bound of the theorem. $\square$

Proof of Theorem 2.6.2. Theorem 2.1 of [19] implies that there is an absolute constant $c > 0$ such that

$$c\,\min\left\{\frac{n^{1/2}}{(1-y^2)^{1/4}}\,,\ n\right\} \leq \sup_{0\not\equiv P\in\mathcal{P}_{n-1}} \frac{|P(y)|}{\|P\|_{L_2[-1,1]}}\,,$$

for every $y \in [-1,1]$. Hence the theorem follows from Remark 1.1. $\square$

Proof of Theorem 2.7.1. Let $f \in \mathcal{T}_n$ be of the form (1.6) with (2.7.1). Let m be an integer such that $n \leq 2m$. We define the entire function of type $\lambda + 2m$ by

$$g(z) := f(z)\left(\frac{\sin z}{z}\right)^{2m}\,.$$

By Bernstein's inequality we have

$$|f'(0)| = |g'(0)| \le (\lambda + 2m) \sup_{t \in \mathbb{R}} |g(t)| \,. \tag{4.8}$$

Lemma 3.2 implies that

$$|g(t)| \le \left(\frac{2et}{2e}\right)^n \|f\|_{[0,2e]} \left(\frac{|\sin t|}{t}\right)^{2m} \le t^{n-2m} \|f\|_{[0,2e]} \le \|f\|_{[0,2e]} \,, \qquad t \ge 2e \,, \tag{4.9}$$

and, as $|\sin t| \le t$ for all $t \ge 0$, obviously

$$|g(t)| \le |f(t)| \,, \qquad t \in [0, 2e] \,. \tag{4.10}$$

Combining (4.9) and (4.10) we have

$$\sup_{t \in [0,\infty)} |g(t)| \le \|f\|_{[0,2e]} \,, \tag{4.11}$$

and similarly

$$\sup_{t \in (-\infty,0]} |g(t)| \le \|f\|_{[-2e,0]} \,. \tag{4.12}$$

Using (4.8), (4.11), and (4.12), we conclude

$$|f'(0)| \le (\lambda + 2m) \, \|f\|_{[-2e,2e]} \,.$$

Transforming the above inequality linearly from the interval $[-2e, 2e]$ to the interval $[-1, 1]$, and choosing m so that $n = 2m$ if n is even, and $n + 1 = 2m$ if n is odd, we get the upper bound of the theorem. To see the sharpness of the upper bound up to the factor $2e$, we pick $f(t) := \sin \lambda t$ if $\lambda \ge n \ge 2$, and $f(t) = T_m(\varepsilon^{-1} \sin(\varepsilon t))$ with a sufficiently small $\varepsilon > 0$, where T_m is the Chebyshev polynomial of degree m defined by $T_m(\cos\theta) = \cos(m\theta)$, $\theta \in [0, 2\pi)$, and m is the largest odd integer such that $2m + 1 \le n$. $\square$

Proof of Theorem 2.8.1. Let $y \in [0, 1]$. Let $f \in \mathcal{T}_n$ be of the form (1.6). Transforming the inequality of Theorem 2.4.1 linearly from the interval $[0, 1]$ to the intervals $[0, y]$ and $[y, 1]$, respectively, we obtain that

$$\begin{aligned} y^3 |f'(y)|^2 &\le 27^2 \, (1 + \varepsilon_n)^2 \, n^3 \left(\sum_{k=1}^n \left(\left(\frac{\lambda_k}{9n} \right)^2 + (k-1)^2 \right) \right) \int_0^y |f(u)|^2 e^{-9n(y-u)/y} \, du \\ &\le 27^2 \, (1 + \varepsilon_n)^2 \, n^3 \left(\sum_{k=1}^n \left(\left(\frac{y\lambda_k}{9n} \right)^2 + (k-1)^2 \right) \right) \frac{y}{9n} \, \|f\|^2_{[0,y]} \,, \end{aligned}$$

and

$$
\begin{aligned}
&(1-y)^3|f'(y)|^2 \\
&\qquad \le (1+\varepsilon_n)^2\, n^3 \left(\sum_{k=1}^{n} \left(\left(\frac{(1-y)\lambda_k}{n} \right)^2 + (k-1)^2 \right) \right) \int_0^y |f(u)|^2 e^{-9n(y-u)/(1-y)}\, du \\
&\qquad \le 27^2\, (1+\varepsilon_n)^2\, n^3 \left(\sum_{k=1}^{n} \left(\left(\frac{(1-y)\lambda_k}{9n} \right)^2 + (k-1)^2 \right) \right) \frac{1-y}{9n} \|f\|^2_{[y,1]} \,.
\end{aligned}
$$

Using the second inequality with $y=0$, we get the first inequality of the theorem. Using the first inequality above if $y \in [1/2, 1]$ and the second inequality above if $y \in [0, 1/2]$ we get

$$
|f'(y)|^2 \le 27^2 (1+\varepsilon_n)^2\, n^3 \left(\sum_{k=1}^{n} \left(\left(\frac{\lambda_k}{9n} \right)^2 + 4(k-1)^2 \right) \right) \frac{1}{9n} \, \|f\|^2_{[0,1]},
$$

and the first statement of the theorem follows. $\square$

Proof of Theorem 2.8.2. Let $Q_n \in \mathcal{P}_n$ be defined by $Q_n(x) = T_n(2x-1)$, where T_n is the Chebyshev polynomial of degree n on $[-1,1]$ defined by $T_n(\cos\theta) = \cos(n\theta)$. As

$$
|P_n'(0)| = 2n^2 = 2n^2 \|P_n\|_{[0,1]} \,,
$$

the theorem follows from Remark 1.1. $\square$

Proof of Theorem 2.9.1. This follows from Lemma 3.12 by the substitution $x = e^{-t}$. $\square$

Proof of Theorem 2.9.2. This follows from Theorem 2.9.1 immediately. $\square$

Proof of Theorem 2.9.3. Let $g \in \mathcal{P}_{n-1}$ be defined by

$$
g(t) = \sum_{j=1}^{n-1} \left(\sin \frac{j\pi}{2n-1} \right) L_j(t) \,,
$$

where L_j is the jth Laguerre polynomial. Associated with $\eta > 0$ we define $f(t) := e^{i\eta t} g(t)$. We have

$$
f'(t) = e^{i\eta t}(f'(t) + i\eta f(t)) \,,
$$

and

$$
\frac{\|f'\|_2}{\|f\|_2} = \left(\eta^2 + \left(2 \sin \frac{\pi}{4n-2} \right)^{-1} \right)^{1/2}
$$

follows from the result of Turán [31] stated in Section 2.9 after Theorem 2.9.3. Using Remark 1.1 we can easily see that there are $0 \not\equiv f_k \in \mathcal{T}_n(\eta)$ such that

$$
\lim_{k\to\infty} \|f_k - f\|_2 = 0 \qquad \text{and} \qquad \lim_{k\to\infty} \|f_k' - f'\|_2 = 0 \,,
$$

and hence

$$\sup_{0 \not\equiv f \in \mathcal{T}_n(\eta)} \frac{\|f'\|_2}{\|f\|_2} \geq \sup_{k \in \mathbb{N}} \frac{\|f_k'\|_2}{\|f_k\|_2} = \left(\eta^2 + \left(2\sin\frac{\pi}{4n-2}\right)^{-1}\right)^{1/2}.$$

□

Proof of Theorem 2.10.1. Observe that if $0 \not\equiv f \in \mathcal{E}_n^-$ is of the form

$$f(t) = \sum_{j=1}^{n} a_j e^{\lambda_j t}, \quad a_j, \lambda_j \in \mathbb{C}\,, \ \ \mathrm{Re}(\lambda_j) < 0\,,$$

then $g \in \mathcal{E}_n$ defined by $g(t) = f(t)e^{t/2}$ is of the form

$$g(t) = \sum_{j=1}^{n} a_j e^{\lambda_j t}, \quad a_j, \lambda_j \in \mathbb{C}\,, \ \ \mathrm{Re}(\lambda_j) < 1/2\,.$$

Now an application of Theorem 2.9.1 to g gives

$$\frac{\left\|\left(\left(f'(t)e^{t/2} + \frac{1}{2}f(t)e^{t/2}\right)\right)e^{-t/2}\right\|_{L_2[0,\infty)}}{\|f(t)e^{t/2}e^{-t/2}\|_{L_2[0,\infty)}} = \frac{\|(g'(t)e^{-t/2}\|_{L_2[0,\infty)}}{\|g(t)e^{-t/2}\|_{L_2[0,\infty)}}$$

$$\leq \max_{1 \leq j \leq n}\left|\lambda_j + \frac{1}{2}\right| + \left(\sum_{j=1}^{n}\left(1 - 2\mathrm{Re}\left(\lambda_j + \frac{1}{2}\right)\right)\sum_{k=j+1}^{n}\left(1 - 2\mathrm{Re}\left(\lambda_k + \frac{1}{2}\right)\right)\right)^{1/2},$$

hence

$$\frac{\|f'\|_{L_2[0,\infty)}}{\|f\|_{L_2[0,\infty)}} \leq \frac{1}{2} + \max_{1 \leq j \leq n}\left|\lambda_j + \frac{1}{2}\right| + 2\left(\sum_{j=1}^{n}\mathrm{Re}(\lambda_j)\sum_{k=j+1}^{n}\mathrm{Re}(\lambda_k)\right)^{1/2}.$$

□

5. Appendix

The paper is self-contained without the results listed in this section. The results below are closely related to our new results in this paper. Theorems 5.1–5.6 have been proved by subtle Descartes system methods which can be employed in the case of exponential sums with only real exponents but not in the case of complex exponents. The reader may find it useful to compare the results in this section with the new results of the paper.

Associated with a set $\Lambda_n := \{\lambda_0, \lambda_1, \dots, \lambda_n\}$ of distinct real numbers let

$$E(\Lambda_n) := \mathrm{span}\{e^{\lambda_0 t}, e^{\lambda_1 t}, \dots, e^{\lambda_n t}\} = \left\{f : f(t) = \sum_{j=0}^{n} a_j e^{\lambda_j t}, \ \ a_j \in \mathbb{R}\right\}.$$

The following result was proved in [15].

Theorem 5.1. *Suppose $\Lambda_n := \{\lambda_0, \lambda_1, \dots, \lambda_n\}$ is a set of distinct nonnegative real numbers. Let $0 < q \le p \le \infty$. Let μ be a non-negative integer. There are constants $c_1 = c_1(p,q,\mu) > 0$ and $c_2 = c_2(p,q,\mu)$ depending only on p, q, and μ such that*

$$c_1 \left(\sum_{j=0}^{n} \lambda_j \right)^{\mu+1/q-1/p} \le \sup_{0 \not\equiv f \in E(\Lambda_n)} \frac{\|f^{(\mu)}\|_{L_p(-\infty,0]}}{\|f\|_{L_q(-\infty,0]}} \le c_2 \left(\sum_{j=0}^{n} \lambda_j \right)^{\mu+1/q-1/p},$$

where the lower bound holds for all $0 < q \le p \le \infty$ and $\mu \ge 0$, while the upper bound holds when $\mu = 0$ and $0 < q \le p \le \infty$, and when $\mu \ge 1$, $p \ge 1$, and $0 < q \le p \le \infty$. Also, there are constants $c_1 = c_1(q,\mu) > 0$ and $c_2 = c_2(q,\mu)$ depending only on q and μ such that

$$c_1 \left(\sum_{j=0}^{n} \lambda_j \right)^{\mu+1/q} \le \sup_{0 \not\equiv f \in E(\Lambda_n)} \frac{|f^{(\mu)}(y)|}{\|f\|_{L_q(-\infty,y]}} \le c_2 \left(\sum_{j=0}^{n} \lambda_j \right)^{\mu+1/q}$$

for all $0 < q \le \infty$, $\mu \ge 1$, and $y \in \mathbb{R}$.

Extending the main result of [1], in [16] we proved the following couple of theorems.

Theorem 5.2. *Suppose $\Lambda_n := \{\lambda_0, \lambda_1, \dots, \lambda_n\}$ is a set of distinct real numbers. Let $0 < q \le p \le \infty$, $a, b \in \mathbb{R}$, and $a < b$. There are constants $c_3 = c_3(p,q,a,b) > 0$ and $c_4 = c_4(p,q,a,b)$ depending only on p, q, a, and b such that*

$$c_3 \left(n^2 + \sum_{j=0}^{n} |\lambda_j| \right)^{1/q-1/p} \le \sup_{0 \not\equiv f \in E(\Lambda_n)} \frac{\|f\|_{L_p[a,b]}}{\|f\|_{L_q[a,b]}} \le c_4 \left(n^2 + \sum_{j=0}^{n} |\lambda_j| \right)^{1/q-1/p}.$$

Theorem 5.3. *Suppose $\Lambda_n := \{\lambda_0, \lambda_1, \dots, \lambda_n\}$ is a set of distinct real numbers. Let $0 < q \le p \le \infty$, $a, b \in \mathbb{R}$, and $a < b$. There are constants $c_5 = c_5(p,q,a,b) > 0$ and $c_6 = c_6(p,q,a,b)$ depending only on p, q, a, and b such that*

$$c_5 \left(n^2 + \sum_{j=0}^{n} |\lambda_j| \right)^{1+1/q-1/p} \le \sup_{0 \not\equiv f \in E(\Lambda_n)} \frac{\|f'\|_{L_p[a,b]}}{\|f\|_{L_q[a,b]}} \le c_6 \left(n^2 + \sum_{j=0}^{n} |\lambda_j| \right)^{1+1/q-1/p},$$

where the lower bound holds for all $0 < q \le p \le \infty$, while the upper bound holds when $p \ge 1$ and $0 < q \le p \le \infty$.

Using the L_∞ norm on a fixed subinterval $[a+\delta, b-\delta] \subset [a,b]$ in the numerator in Theorem 5.2, we proved the following essentially sharp result in [7]. For the sake of brevity let

$$\|f\|_A := \sup_{t \in A} |f(t)|$$

for a complex-valued function f defined on a set $A \subset \mathbb{R}$.

Theorem 5.4. *Let $a, b \in \mathbb{R}$, $a < b$. If $\Lambda_n := \{\lambda_0, \lambda_1, \dots, \lambda_n\}$ is a set of distinct real numbers, then the inequality*

$$\|f\|_{[a+\delta, b-\delta]} \le e8^{1/p} \left(\frac{n+1}{\delta} \right)^{1/p} \|f\|_{L_p[a,b]}$$

holds for every $f \in E(\Lambda_n)$, $p > 0$ and $\delta \in \left(0, \frac{1}{2}(b-a)\right)$.

The key to Theorem 5.4 is the following Remez-type inequality proved also in [7]. For the sake of brevity let

$$E_n := \left\{ f : f(t) = a_0 + \sum_{j=1}^{n} a_j e^{\lambda_j t}, \ \ a_j, \lambda_j \in \mathbb{R} \right\}$$

and

$$E_n(s) := \{ f \in E_n : m(\{x \in [-1,1] : |f(x)| \le 1\}) \ge 2 - s \},$$

where $m(A)$ denotes the Lebesgue measure of a measurable set $A \subset \mathbb{R}$.

Theorem 5.5. *Let $s \in \left(0, \frac{1}{2}\right]$. There are absolute constants $c_7 > 0$ and $c_8 > 0$ such that*

$$\exp(c_7 \min\{ns, (ns)^2\}) \le \sup_{f \in E_n(s)} |f(0)| \le \exp(c_8 \min\{ns, (ns)^2\}).$$

An essentially sharp Bernstein-type inequality for E_n is proved in [5].

Theorem 5.6. *Let $a, b \in \mathbb{R}$, $a < b$. We have*

$$\frac{1}{e-1} \, \frac{n-1}{\min\{y-a, b-y\}} \le \sup_{0 \not\equiv f \in E_n} \frac{|f'(y)|}{\|f\|_{[a,b]}} \le \frac{2n-1}{\min\{y-a, b-y\}}, \qquad y \in (a,b).$$

Having real exponents λ_j in Theorems 5.1–5.6 is essential in the proofs using subtle Descartes system methods. There are other important inequalities proved for the classes $E(\Lambda_n)$ associated with a set $\Lambda_n := \{\lambda_0, \lambda_1, \dots, \lambda_n\}$ of distinct real exponents; see [6], for instance, where the proofs are using Descartes system methods as well.

Let V_n be a vector space of complex-valued functions defined on $\mathbb{R}$ of dimension $n+1$ over $\mathbb{C}$. We say that V_n is shift invariant (on $\mathbb{R}$) if $f \in V_n$ implies that $f_a \in V_n$ for every $a \in \mathbb{R}$, where $f_a(x) := f(x-a)$ on $\mathbb{R}$. Associated with a set $\Lambda_n := \{\lambda_0, \lambda_1, \dots, \lambda_n\}$ of distinct COMPLEX numbers let

$$E^c(\Lambda_n) := \operatorname{span}\{e^{\lambda_0 t}, e^{\lambda_1 t}, \dots, e^{\lambda_n t}\} = \left\{ f : f(t) = \sum_{j=0}^{n} a_j e^{\lambda_j t}, \ \ a_j \in \mathbb{C} \right\}.$$

Elements of $E^c(\Lambda_n)$ are called exponential sums of $n+1$ terms. Examples of shift invariant spaces of dimension $n+1$ include $E^c(\Lambda_n)$. In [8] we proved a result analogous to Theorem 5.4 for exponential sums with complex exponents λ_j, in which case Descartes system methods cannot help us in the proof.

Theorem 5.7. *Let $a, b \in \mathbb{R}$, $a < b$. Let $V_n \subset C[a,b]$ be a shift invariant vector space of complex-valued functions defined on $\mathbb{R}$ of dimension $n+1$ over $\mathbb{C}$. We have*

$$\|f\|_{[a+\delta, b-\delta]} \leq 2^{2/p^2} \left(\frac{n+1}{\delta} \right)^{1/p} \|f\|_{L_p[a,b]}$$

for every $f \in V_n$, $p \in (0,2]$, and $\delta \in \left(0, \frac{1}{2}(b-a)\right)$, and

$$\|f\|_{[a+\delta, b-\delta]} \leq 2^{1/2} \left(\frac{n+1}{\delta} \right)^{1/2} (b-a)^{(p-2)/p} \|f\|_{L_p[a,b]}$$

for every $f \in V_n$, $p \geq 2$, and $\delta \in \left(0, \frac{1}{2}(b-a)\right)$.

It is well known by considering the case of algebraic polynomials of degree n that, in general, the size of the factor $(n+1)^{1/p}$ in Theorem 5.7 cannot be improved for $p \in (0,2]$. On the other hand, for $p \geq 2$ the size of the factor $(n+1)^{1/2}$ in the inequality

$$\begin{aligned}
\|f\|_{[a+\delta, b-\delta]} &\leq 2^{1/2} \left(\frac{n+1}{\delta} \right)^{1/2} \|f\|_{L_2[a,b]} \\
&\leq 2^{1/2} \left(\frac{n+1}{\delta} \right)^{1/2} (b-a)^{(p-2)/(2p)} \|f\|_{L_p[a,b]}
\end{aligned}$$

cannot be improved. This can be seen by taking lacunary trigonometric polynomials; see the theorem below from [33, p. 215].

Theorem 5.8. *Let (k_j) be a strictly increasing sequence of nonnegative integers satisfying*

$$k_{j+1} > \alpha k_j \,, \qquad j = 1, 2, \dots \,,$$

where $\alpha > 1$. Let

$$Q_n(t) = \sum_{j=1}^{n} \cos(2\pi k_j (t - \theta_{j,n})) \,, \qquad \theta_{j,n} \in \mathbb{R} \,.$$

There are constants $A_{q,\alpha} > 0$ and $B_{q,\alpha} > 0$ depending only on q and α such that

$$A_{q,\alpha} n^{1/2} \leq \|Q_n\|_{L_q[0,1]} \leq B_{q,\alpha} n^{1/2}$$

for every $n \in \mathbb{N}$ and $q > 0$.

6. Acknowledgements

The author wishes to thank Sergey Denisov for the motivating questions and e-mail discussions related to the paper. The author wishes to thank Stephen Choi as well for reading earlier versions of my paper carefully, pointing out many misprints, and his advise about the structure of the presentation.

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