THE NORM OF THE POLYNOMIAL TRUNCATION OPERATOR ON THE UNIT DISK AND ON $[-1,1]$

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Abstract. Let D and ∂D denote the open unit disk and the unit circle of the complex plane, respectively. We denote the set of all polynomials of degree at most n with real coefficients by $\mathcal{P}_n$. We denote the set of all polynomials of degree at most n with complex coefficients by $\mathcal{P}_n^c$. We define the truncation operator S_n for polynomials $P_n \in \mathcal{P}_n^c$ of the form

$$P_n(z) := \sum_{j=0}^{n} a_j z^j\,, \qquad a_j \in \mathbb{C}\,,$$

by

$$S_n(P_n)(z) := \sum_{j=0}^{n} \widetilde{a}_j z^j\,, \qquad \widetilde{a}_j := (a_j/|a_j|)\min\{|a_j|,1\} \tag{1.1}$$

(here $0/0$ is interpreted as 1). We define the norms of the truncation operators by

$$\|S_n\|_{\infty,\partial D}^{real} := \sup_{P_n \in \mathcal{P}_n} \frac{\max_{z\in\partial D}|S_n(P_n)(z)|}{\max_{z\in\partial D}|P_n(z)|}$$

and

$$\|S_n\|_{\infty,\partial D}^{comp} := \sup_{P_n \in \mathcal{P}_n^c} \frac{\max_{z\in\partial D}|S_n(P_n)(z)|}{\max_{z\in\partial D}|P_n(z)|}\,.$$

Our main theorem in this paper establishes the right order of magnitude of the norms of the operators S_n. This settles a question asked by S. Kwapien.

Theorem. *With the notation introduced above there is an absolute constant $c_1 > 0$ such that*

$$c_1\sqrt{2n+1} \leq \|S_n\|_{\infty,\partial D}^{real} \leq \|S_n\|_{\infty,\partial D}^{comp} \leq \sqrt{2n+1}\,.$$

Moreover, an analogous result in $L_p(\partial D)$ for $p \in [2,\infty]$ is also established and the case when the unit circle ∂D is replaced by the interval $[-1,1]$ is also studied.

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1. New Result

Let D and ∂D denote the open unit disk and the unit circle of the complex plane, respectively. We denote the set of all polynomials of degree at most n with real coefficients by $\mathcal{P}_n$. We denote the set of all polynomials of degree at most n with complex coefficients by $\mathcal{P}_n^c$. We define the truncation operator S_n for polynomials $P_n \in \mathcal{P}_n^c$ of the form

$$P_n(z) := \sum_{j=0}^{n} a_j z^j \,, \qquad a_j \in \mathbb{C}\,,$$

by

$$S_n(P_n)(z) := \sum_{j=0}^{n} \widetilde{a}_j z^j \,, \qquad \widetilde{a}_j := (a_j/|a_j|)\min\{|a_j|, 1\} \tag{1.1}$$

(here $0/0$ is interpreted as 1). In other words, we take the coefficients $a_j \in \mathbb{C}$ of a polynomial P_n of degree at most n, and we truncate them. That is, we leave a coefficient a_j unchanged if $|a_j| < 1$, while we replace it by $a_j/|a_j|$ if $|a_j| \geq 1$. We form the new polynomial with the new coefficients $\widetilde{a}_j$ defined by (1.1), and we denote this new polynomial by $S_n(P_n)$. We define the norms of the truncation operators by

$$\|S_n\|_{\infty,\partial D}^{real} := \sup_{P_n \in \mathcal{P}_n} \frac{\max_{z\in\partial D} |S_n(P_n)(z)|}{\max_{z\in\partial D} |P_n(z)|}$$

and

$$\|S_n\|_{\infty,\partial D}^{comp} := \sup_{P_n \in \mathcal{P}_n^c} \frac{\max_{z\in\partial D} |S_n(P_n)(z)|}{\max_{z\in\partial D} |P_n(z)|}\,.$$

Our main theorem in this paper establishes the right order of magnitude of the norms of the operators S_n. This settles a question asked by S. Kwapien.

Theorem 1.1. *With the notation introduced above there is an absolute constant $c_1 > 0$ such that*

$$c_1\sqrt{2n+1} \leq \|S_n\|_{\infty,\partial D}^{real} \leq \|S_n\|_{\infty,\partial D}^{comp} \leq \sqrt{2n+1}\,.$$

In fact we are able to establish an $L_p(\partial D)$ analogue of this as follows. For $p \in (0,\infty)$, let

$$\|S_n\|_{p,\partial D}^{real} := \sup_{P_n \in \mathcal{P}_n} \frac{\|S_n(P_n)\|_{L_p(\partial D)}}{\|P_n\|_{L_p(\partial D)}}$$

and

$$\|S_n\|_{p,\partial D}^{comp} := \sup_{S_n \in \mathcal{P}_n^c} \frac{\|S_n(P_n)\|_{L_p(\partial D)}}{\|P_n\|_{L_p(\partial D)}}\,.$$

Theorem 1.2. *With the notation introduced above there is an absolute constant $c_1 > 0$ such that*

$$c_1(2n+1)^{1/2-1/p} \leq \|S_n\|_{p,\partial D}^{real} \leq \|S_n\|_{p,\partial D}^{comp} \leq (2n+1)^{1/2-1/p}$$

for every $p \in [2, \infty)$.

Note that it remains open what is the right order of magnitude of $\|S_n\|_{p,\partial D}^{real}$ and $\|S_n\|_{p,\partial D}^{comp}$, respectively when $0 < p < 2$. In particular it would be interesting to see if $\|S_n\|_{p,\partial D}^{comp} \leq c$ is possible for any $0 < p < 2$ with an absolute constant c. We record the following observation, due to S. Kwapien, in this direction.

Theorem 1.3. *There is an absolute constant $c > 0$ such that*

$$\|S_n\|_{1,\partial D}^{real} \geq c\sqrt{\log n}\,.$$

If the unit circle ∂D is replaced by the interval $[-1, 1]$, we get a completely different order of magnitude of the polynomial truncation projector. In this case the norms of the truncation operators S_n are defined in the usual way. That is, let

$$\|S_n\|_{\infty,[-1,1]}^{real} := \sup_{P_n \in \mathcal{P}_n} \frac{\max_{x\in[-1,1]} |S_n(P_n)(x)|}{\max_{x\in[-1,1]} |P_n(x)|}$$

and

$$\|S_n\|_{\infty,[-1,1]}^{comp} := \sup_{P_n \in \mathcal{P}_n^c} \frac{\max_{x\in[-1,1]} |S_n(P_n)(x)|}{\max_{x\in[-1,1]} |P_n(x)|}\,.$$

Theorem 1.4. *With the notation introduced above we have*

$$2^{n/2-1} \leq \|S_n\|_{\infty,[-1,1]}^{real} \leq \|S_n\|_{\infty,[-1,1]}^{comp} \leq \sqrt{2n+1} \cdot 8^{n/2}\,.$$

2. Lemmas

To prove the lower bound of Theorem 1.1 and 1.2, we need two lemmas. Our first one is from [LSV].

Lemma 2.1 (Lovász, Spencer, Vesztergombi). *Let $a_{j,k}, j = 1, 2, \ldots, n_1, k = 1, 2, \ldots, n_2$ be such that $|a_{j,k}| \leq 1$. Let also $p_1, p_2, \ldots, p_{n_2} \in [0, 1]$. Then there are choices*

$$\varepsilon_k \in \{-p_k, 1-p_k\}, \qquad k = 1, 2, \ldots, n_2\,,$$

such that for all j,

$$\left|\sum_{k=1}^{n_2} \varepsilon_k a_{j,k}\right| \leq C\sqrt{n_1}$$

with an absolute constant C.

Our second lemma is a direct consequence of the well-known Bernstein inequality (see Theorem 1.1 on page 97 of [DL]) and the Mean Value Theorem.

Lemma 2.2. *Suppose Q_n is a polynomial of degree n (with complex coefficients),*

$$\theta_n := \exp\left(\frac{2\pi}{14n}\right),$$

$$z_j := \exp(ij\theta_n), \qquad j = 1, 2, \dots, 14n,$$

and

$$|Q_n(z_j)| \leq M, \qquad j = 1, 2, \dots, 14n.$$

Then

$$\max_{z \in \partial D} |Q_n(z)| \leq 2M.$$

The inequalities below (see Theorem 2.6 on page 102 of [DL]) will be needed to prove the upper bound of the Theorem.

Lemma 2.3 (Nikolskii Inequality). *Let $0 < q \leq p \leq \infty$. If P_n is a polynomial of degree at most n with complex coefficients then*

$$\|P_n\|_{L_p(\partial D)} \leq \left(\frac{2nr+1}{2\pi}\right)^{1/q-1/p} \|P_n\|_{L_q(\partial D)},$$

where $r = r(q)$ is the smallest integer not less than $q/4$.

The next lemma may be found in [Ri].

Lemma 2.4 (Erdős). *Suppose that $z_0 \in \mathbb{C}$ and $|z_0| \geq 1$. Then*

$$|P_n(z_0)| \leq |T_{2n}(z_0)|^{1/2} \cdot \max_{x \in [-1,1]} |P_n(x)|, \qquad P_n \in \mathcal{P}_n^c,$$

where $T_{2n} \in \mathcal{P}_{2n}$ defined by

$$T_n(x) := \cos(n \arccos x), \qquad x \in [-1, 1],$$

is the Chebyshev polynomial of degree $2n$. As a consequence, writing

$$T_{2n}(z) = 2^{2n-1} \prod_{j=1}^{n} (z^2 - x_j^2), \qquad x_j \in (0, 1),$$

we have

$$\max_{z \in \partial D} |P_n(z)| \leq 8^{n/2} \max_{x \in [-1,1]} |P_n(x)|.$$

3. Proof of the theorem

Proof of Theorem 1.1. We apply Lemma 2.1 with $n_1 = 14n, n_2 = n$,

$$\theta_n := \exp(2\pi/(14n))\,, \qquad a_{j,k} := \exp(ijk\theta_n)\,,$$

and $p_1 = p_2 = \cdots = p_n = 1/3$, and with the choices

$$\varepsilon_k \in \left\{-\frac{1}{3}, \frac{2}{3}\right\}, \qquad k = 1, 2, \ldots, n\,,$$

coming from Lemma 2.1, we define

$$Q_n(z) = 3\sum_{j=1}^{n} \varepsilon_k z^k\,.$$

Then Q_n is a polynomial of degree n with each coefficient in $\{-1, 2\}$, and with the notation

$$z_j := \exp(ij\theta_n)\,, \qquad j = 1, 2, \ldots, 14n\,,$$

we have

$$|Q_n(z_j)| \le 3C\sqrt{14n}\,, \qquad j = 1, 2, \ldots, 14n\,.$$

Hence Lemma 2.2 yields

$$\max_{z\in\partial D} |Q_n(z)| \le 24C\sqrt{n} \tag{3.1}$$

In particular, if we denote by m the number of indices k for which $\varepsilon_k = 2/3$, then

$$|3m - n| = |2m - (n - m)| = |Q_n(1)| \le 24C\sqrt{n}\,,$$

hence

$$|S_n(Q_n)(1)| = |m - (n - m)| = |2m - n| \ge \frac{n}{3} - 32C\sqrt{n}\,. \tag{3.2}$$

Now (3.1) and (3.2) give the lower bound of the theorem.

To see the upper bound of the theorem, observe that Lemma 2.3 implies

$$\begin{aligned}\max_{z\in\partial D} |S_n(P_n)(z)| &\le \frac{\sqrt{2n+1}}{\sqrt{2\pi}}\|S_n(P_n)\|_{L_2(\partial D)} \le \frac{\sqrt{2n+1}}{\sqrt{2\pi}}\|P_n\|_{L_2(\partial D)} \\ &\le \sqrt{2n+1}\max_{z\in\partial D}|P_n(z)|\end{aligned}$$

for all polynomials P_n of degree at most n with complex coefficients. This proves the upper bound of the theorem. □

Proof of Theorem 1.2. Let $p \in [2, \infty)$. Using (3.2) and the Nikolskii-type inequality of Lemma 3.4, we obtain that

$$\|S_n(Q_n)\|_{L_p(\partial D)} \ge c_1 n^{1-1/p} \tag{3.3}$$

with an absolute constant $c_1 > 0$. On the other hand (3.1) implies

$$\|Q_n\|_{L_p(\partial D)} \leq c_2 n^{1/2} \tag{3.4}$$

with an absolute constant $c_2 > 0$, and the lower bound of the theorem follows.

To see the upper bound of the theorem, observe that Lemma 2.3 implies

$$\begin{aligned}
\|S_n(P_n)\|_{L_p(\partial D)} &\leq \left(\frac{2n+1}{2\pi}\right)^{1/2-1/p} \|S_n(P_n)\|_{L_2(\partial D)} \\
&\leq \left(\frac{2n+1}{2\pi}\right)^{1/2-1/p} \|P_n\|_{L_2(\partial D)} \\
&\leq (2n+1)^{1/2-1/p} \|P_n\|_{L_p(\partial D)}
\end{aligned}$$

for all polynomials P_n of degree at most n with complex coefficients. This proves the upper bound of the theorem. $\square$

Proof of Theorem 1.3. Let $n = 2^{m+2} - 2$. Consider the polynomial

$$P_n(z) = 4z^{2^{m+1}-1} \prod_{k=0}^{m} \left(1 + \frac{z^{2^k} + z^{-2^k}}{2}\right).$$

Then

$$|P_n(z)| = 4 \prod_{k=0}^{m} \left(1 + \frac{z^{2^k} + z^{-2^k}}{2}\right),$$

and hence $\|P_n\|_{L_1(\partial D)} = 4$. Also

$$P_n(z) - S_n(P_n)(z) = z^{2^{m+1}-1} \left(3 + \sum_{k=0}^{m} \left(z^{2^k} + z^{-2^k}\right)\right).$$

Let

$$R_n(z) := 3 + \sum_{k=0}^{m} \left(z^{2^k} + z^{-2^k}\right).$$

Then

$$\|S_n(P_n)\|_{L_1(\partial D)} \geq \|S_n(P_n) - P_n\|_{L_1(\partial D)} - \|P_n\|_{L_1(\partial D)} = \|R_n\|_{L_1(\partial D)} - 4\,.$$

We will prove that $\|R_n\|_{L_1(\partial D)} \geq c\sqrt{m}$ for some absolute constant $c > 0$. It is easy to see that if $b, a_0, a_1, \ldots, a_m$ are complex numbers and

$$F(z) = b + \sum_{k=0}^{m} a_k \left(z^{2^k} + z^{-2^k}\right),$$

then we have

$$\|F\|_{L_4(\partial D)} \leq \sqrt[4]{3} \left(|b|^2 + \sum_{k=0}^{m} |2a_k|^2\right)^{1/2}.$$

Therefore
$$\|R_n\|_{L_4(\partial D)} \leq \sqrt[4]{3}\sqrt{9+4(m+1)}\,.$$
Moreover
$$\|R_n\|_{L_2(\partial D)} = \sqrt{9+2(m+1)}\,.$$
By Hölder's Inquality
$$\|R_n\|^{2/3}_{L_4(\partial D} \|R_n\|^{1/3}_{L_1(\partial D)} \geq \|R_n\|_{L_2(\partial D)}\,.$$
Hence we obtain
$$\left(\sqrt[4]{3}\sqrt{9+4(m+1)}\right)^{2/3} \left(\|R_n\|_{L_1(\partial D)}\right)^{1/3} \geq \sqrt{9+2(m+1)}\,,$$
and thus $\|R_n\|_{L_1(\partial D)} \geq c\sqrt{m}$. This gives
$$\frac{\|S_n(P_n)\|_{L_1(\partial D)}}{\|P_n\|_{L_1(\partial D)}} \geq c'\sqrt{m} \geq c''\sqrt{\log n}$$
with absolute constants $c' > 0$ and $c'' > 0$. □

Proof of Theorem 1.4. Let
$$P_n(z) := \sum_{j=0}^{n} a_j z^j\,, \qquad a_j \in \mathbb{C}\,,$$
and
$$S_n(P_n)(z) := \sum_{j=0}^{n} \widetilde{a}_j z^j\,, \qquad \widetilde{a}_j := (a_j/|a_j|)\min\{|a_j|, 1\}\,.$$
First we prove the upper bound. Using Lemma 1.4 we obtain
$$\begin{aligned}\max_{x\in[-1,1]} |S_n(P_n)(x)| &\leq \max_{z\in\partial D} |S_n(P_n)(z)| \\ &\leq \left(\frac{2n+1}{2\pi}\right)^{1/2} \|S_n(P_n)\|_{L_2(\partial D)} \\ &\leq \left(\frac{2n+1}{2\pi}\right)^{1/2} \|P_n\|_{L_2(\partial D)} \\ &\leq \left(\frac{2n+1}{2\pi}\right)^{1/2} \cdot 8^{n/2}\sqrt{2\pi} \max_{x\in[-1,1]} |P_n(x)|\,,\end{aligned}$$
which proves the upper bound of the theorem.

Now we turn to the lower bound. We define $Q_n \in \mathcal{P}_{4n}$ by
$$Q_n(z) := z^{2n}(1-z^2)^n = z^{2n}\sum_{j=0}^{n} (-1)^j \binom{n}{j} z^{2j}\,.$$

Then

$$\max_{x\in[-1,1]} |Q_n(x)| = \left(\frac{1}{4}\right)^n . \tag{3.5}$$

Also

$$S_n(Q_n)(z) = z^{2n} \sum_{j=0}^{n} (-1)^j z^{2j} ,$$

hence for every positive even n

$$|S_n(Q_n)(1)| = 1 . \tag{3.6}$$

Now we conclude the the lower bound of the theorem by combining (3.5) and (3.6). □

4. Acknowledgment.

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