Projective and affine equivalence of sub-Riemannian metrics: integrability, generic rigidity, the Weyl type theorems, and separation of variables (the de Rham type decomposition) conjecture

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based on the joint work with Frederic Jean (ENSTA, Paris) and Sofya Maslovskaya (INRIA, Sophia Antipolis)

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On projective and affine Equivalence of Riemannian metrics

Definition

Two Riemannian metrics g_1 and g_2 on a manifold M are called projectively equivalent if they have the same geodesics , up to a reparametrization.

They are called affinely equivalent, if they have the same geodesics , up to an affine reparametrization.

Two Riemannian metrics are affinely equivalent if and only if they have the same Levi-Civita connection.

Notation: $g_1 \overset{p}{\sim} g_2$ and $g_1 \overset{a}{\sim} g_2$, respectively.

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Projective and affine rigidity and an example of a nonrigid metric

Obviously $g_1 \overset{a}{\sim} Cg_1$ for a positive constant C (we say that Cg_1 is constantly proportional to g_1).

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A metric on a connected manifold M is called ***projectively (affinely) rigid***, *if constantly proportional metrics are the only metrics which are projectively (affinely) equivalent to it.*

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Example of a projectively nonrigid metric

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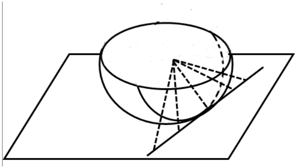
The flat metric is not projectively rigid: If g_1 *is the flat metric on a plane,* g_2 *is a standard metric on a hemisphere, and* F *is the stereographic projection from the center of the hemisphere to the plane (the gnomonic map projection) , then* $(F^{-1})^* g_2 \sim g_1$ *but they are not constantly proportional.*



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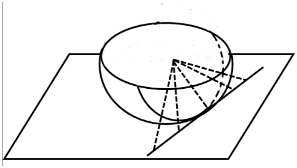

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Transition operator and stabiliity.

All pairs of locally projectively equivalent Riemannian metrics with certain regularity assumption were described by Levi-Civita (1898), generalizing the previous result of Dini of 1869 on 2-dimensional case. These results exhibit certain separation of variables phenomenon.

Given two Riemannian metrics g_1 and g_2 let $S_q : T_qM \mapsto T_qM$ satisfy

$$g_{2_q}(v_1, v_2) = g_{1_q}(S_q v_1, v_2), \quad v_1, v_2 \in T_qM.$$

S_q is called the transition operator from the metrics g_1 to the metrics g_2 at the point q.

S_q is self-adjoint w.r.t. the Euclidean structure given by g_1.

A point $q_0 \in M$ is called stable w.r.t. the pair (g_1, g_2) if S_q has the same number of distinct eigenvalues in a neighborhood of q_0 (or, equivalently, the tuple of multiplicities of the eigenvalues is constant in a neighborhood of q_0).

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Levi-Civita pairs of metrics

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We say that two Riemannian metrics g_1 and g_2 constitute a Levi-Civita pair at a point q_0 if there exist positive integers $k_1, \dots k_m$, $\sum k_s = \dim M$, a local coordinate system $\bar{x} = (\bar{x}_1, \dots, \bar{x}_m)$, where $\bar{x}_s = (x_s^1, \dots, x_s^{k_s})$, and $\forall s, 1 \le s \le m$ a Riemannian metric b_s and a function β_s, both depending on variables $\bar{x}_s$ only and with β_s being constant if $k_s > 1$ and $\beta_s(q_0) \neq \beta_l(q_0)$ for all $s \neq l$, so that

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Levi-Civita and Eisenhart theorems

Theorem (Levi-Civita, 1898)

$g_1 \overset{p}{\sim} g_2$ in a neighborhood of a stable point $q_0 \in M \Leftrightarrow$ if and only g_1 and g_2 form a Levi-Civita pair at q_0.

Theorem (Eisenhart, 1923 for affine case)

$g_1 \overset{a}{\sim} g_2$ in a neighborhood of a stable point $q_0 \in M \Leftrightarrow$ if and only if g_1 and g_2 form a Levi-Civita pair at q_0 such that all functions β_i are constant.

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Case dim M=2 (Dini, 1869)

There exist local coordinates (x, y)

$$g_1 = (X(x) - Y(y))\,(dx^2 + dy^2)$$

$$g_2 = \left(\frac{1}{Y(y)} - \frac{1}{X(x)}\right)\left(\frac{dx^2}{X(x)} + \frac{dy^2}{Y(y)}\right).$$

Liouville surfaces

Existence of nontrivial quadratic integrals

Levi-Civita also showed that, in addition to the kinetic energy integral, the geodesic flow of g_1 admits $m-1$ integrals which are quadratic with respect to velocities (all these m integrals are in involution).

In particular, if $m>1$ it admits the following integral:

$$\left(\prod_{s=1}^{m}\lambda_s\right)^{-\frac{2}{m+1}} g_2(\dot{x},\dot{x})$$

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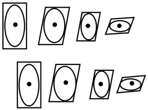
Sub-Riemannian metrics

A rank ℓ distribution $D = \{D(q)\}_{q \in M}$ on a manifold M is a rank ℓ subbundle of the tangent bundle TM (a smooth field of ℓ-dimensional subspaces $D(q)$ of the tangent spaces T_qM).
D is called bracket-generating distribution if at any point iterated Lie brackets of vector fields tangent to D generate the whole tangent space.
Rashevsky-Chow Any two points of M can be connected by a curve tangent to a distribution.

A sub-Riemannian metric g is given on the distribution D, if an inner product g_q is chosen on each subspaces $D(q)$ smoothly in q.

Riemannian case: $D = TM$



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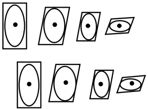

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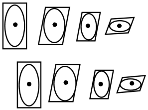

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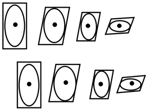

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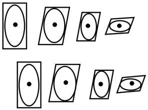

Sub-Riemannian geodesics

Given a sub-Riemannian metric g, for any curve γ tangent to the distribution one can define the sub-Riemannian length by $\int g\big(\dot\gamma(t),\dot\gamma(t)\big)^{\frac{1}{2}}dt$.

Sub-Riemannian geodesics are the candidates for length-minimizers (via the Pontryagin Maximum Principle in Optimal Control).
Two types of geodesics:

- Abnormal -depend on the distribution D but not on the metric as unparametrized curves (no such geodesics in Riemannian case).
- Normal-projections to M of integral curves of the Hamiltonian system on T^*M corresponding to the Hamiltonian $h(p,q)=\frac{1}{2}||p|_{D(q)}||^2$ lying on the level set $h=\frac{1}{2}$ (in the Riemannian case these are exactly Riemannian geodesics).

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Definition

Two sub-Riemannian metrics g_1 and g_2 on a distribution D are called projectively/affinely equivalent if they have the same normal geodesics, up to a reparametrization/an affine parametrization.

Sub-Riemannian geodesics

Given a sub-Riemannian metric g, for any curve γ tangent to the distribution one can define the sub-Riemannian length by $\int g\big(\dot\gamma(t),\dot\gamma(t)\big)^{\frac{1}{2}}dt$.

Sub-Riemannian geodesics are the candidates for length-minimizers (via the Pontryagin Maximum Principle in Optimal Control).

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Distributions admitting product structure

Construction of pairs of projectively equivalent sub-Riemannian metrics by analogy with the metrics appearing in the Levi-Civita theorem:

Let $n = \dim M$. Fix positive integers $k_1, k_2, \dots k_m$ such that $n = k_1 + k_2 + \dots + k_m$. Let $\bar{x}_s = (x_s^1, \dots, x_s^{k_s})$ and $\bar{x} = (\bar{x}_1, \dots, \bar{x}_m)$ are standard coordinates in $\mathbb{R}^n = \mathbb{R}^{k_1} \times \mathbb{R}^{k_2} \times \dots \mathbb{R}^{k_m}$, where $\mathbb{R}^{k_s}$ has standard coordinates $\bar{x}_s$.

For any $1 \leq s \leq m$ let D_s be a bracket generating distribution in $\mathbb{R}^{k_s}$.

Consider the distribution D on $\mathbb{R}^n$ which is obtained by the product of distributions D_s.

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where the velocities $\dot{\bar{x}}$ belong to D, $\lambda_s(\bar{x}) = \beta_s(\bar{x}_s) \prod_{l=1}^{m} \beta_l(\bar{x}_l)$, $\gamma_s(\bar{x}) = \prod_{l \neq s} \left| \frac{1}{\beta_l(\bar{x}_l)} - \frac{1}{\beta_s(\bar{x}_s)} \right|$.
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The case of corank 1 distributions

Conjecture *The generalized Levi-Civita pairs are the only pairs of locally projectively equivalent sR metrics under certain regularity assumptions.*

The answer yet is known to be positive for several cases beyond the Riemannian one (in the sequel we assume the stability of the transition operator):

- sR metrics on contact distributions (I. Z., 2006). In this case it means that any sR metric is projectively rigid, because D does not admit product structure;
- sR metrics on quasi-contact distributions (I. Z. 2006). Generic sR metrics are projectively rigid;
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Conformally and Weyl projectively rigidity

Definition

A sR metric g_1 is called conformally projectively rigid if $g_2 \overset{p}{\sim} g_1$ implies that g_2 is conformal to g_1.

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A sR metric g is said to be Weyl projectively rigid if any metric, which is simultaneously conformal to g and projectively equivalent to g is constantly proportional to g.

Theorem (Weyl 1921; Levi-Civita's Thm with spectral size 1)

For $\dim M > 1$ any Riemannian metric is Weyl projectively rigid.

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Normal sub-Riemannian geodesics are projections to M of integral curves of the Hamiltonian system on T^*M corresponding to the sR Hamiltonian $h(p,q) = \frac{1}{2}||p|_{D(q)}||^2$ lying on the level set $h = \frac{1}{2}$. The integral curves of this Hamiltonian system are called normal extremals. The sub-Riemannian Hamiltonian is trivially an integral of the flow of normal extremals.

Theorem (Geom. Dedicata, 2019, arXiv:1801.04257v2)

If a sub-Riemannian metric is not conformally rigid, then the flow of its normal extremals admits a nontrivial integral quadratic in impulses (i.e. on the fibers of T^*M), namely the integral of Painlevè type.

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Theorem (Geom. Dedicata, 2019, arXiv:1801.04257v2)

If a sub-Riemannian metric is not conformally rigid, then the flow of its normal extremals admits a nontrivial integral quadratic in impulses (i.e. on the fibers of T^*M), namely the integral of Painlevè type.

Corollary

Generic sub-Riemannian metrics on a given distribution are conformally projectively rigid and therefore affinely rigid (and actually projectively rigid in real analytic category).

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Tanaka symbol and nilpotent approximation of a distribution

D is called equiregular at q_0 if all D^j have constant dimension in a neighborhood of q_0.

Definition

- *The (Tanaka) symbol of an equiregular distribution D at a point q_0 is the graded nilpotent Lie algebra $D(q_0) \oplus D^2(q_0)/D(q_0) \oplus D^3(q_0)/D^2(q_0) \oplus \cdots$.*
- *The left-invariant distribution on the corresponding Lie group obtained by the left translation of $D(q_0)$ is called the nilpotent approximation of D at q_0 and is denote by $\widehat{D}_{q_0}$.*

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- *The symbol of an sR metric g is the pair consisting of the Tanaka symbol of D at q_0 and the Euclidean structure $g(q_0)$ on $D(q_0)$.*
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Direct product structure on the level of nilpotent approximation

Theorem (Geom. Dedicata, 2019, arXiv:1801.04257v2)

If g_1 and g_2 are two sub-Riemannian metric on an equiregular distribution D, which are locally projectively equivalent around a stable point q_0 and not conformal, then the nilpotent approximation $\hat{D}_{q_0}$ of D at q_0 admits a product structure and the corresponding nilpotent approximations $\hat{g}_1$ and $\hat{g}_2$ form a Levi-Civita pair with constant coefficients.

Corollary

Any sub-Riemannian metric on a rank 2 bracket generating distribution is affinely rigid and conformally projectively rigid.

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Genericity of indecomposable fundamental graded Lie algebras

Let $\mathrm{GNLA}(m, n)$ be the set of all n-dimensional negatively graded Lie algebras generated by the homogeneous component of weight -1 and such that this component has dimension m.

Proposition

Except the following two cases:

1. $m = n - 1$ *with even* n,
2. $(m, n) = (4, 6)$,

a generic element of $\mathrm{GNLA}(m, n)$ *cannot be represented as a direct sum of two graded Lie algebras.*

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Rigidity of SR structures on generic distribution

Theorem (Geom. Dedicata, 2019, arXiv:1801.04257v2)

Let m and n be two integers such that $2 \le m < n$, and assume $(m, n) \ne (4, 6)$ and $m \ne n - 1$ if n is even. Then, given an n-dimensional manifold M and a generic rank m distribution D on M, any sub-Riemannian metric on (M, D) conformally projectively rigid and therefore affinely rigid (and in the real analytic category even projectively rigid from the following sub-Riemannian Weyl results).

Theorem (preprint, arXiv:2001.08584)

Let m and n be two integers such that $2 \le m < n$. On a generic real analytic rank m distribution D on a connected n-dimensional real analytic manifold M any sub-Riemannian metric is Weyl projectively rigid.

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Decoupling/direct product on the level of Jacobi equations/Jacobi curves of extremals

Projective/affine equivalence of g_1 and g_2 (with Hamiltonians h_1 and h_2) $\Rightarrow$ existence of the fiber-preserving preserving orbital diffeomorphism Φ between Hamiltonian flows on an open dense sets of the cotangent bundle, i.e.

$$\Phi_* \vec{h}_1 = a\vec{h}_2 \text{ on an open set of } T^*M.$$

Theorem (I.Z.)

If a sub-Riemannian metric is not affinely rigid then the Jacobi equation along generic normal extremal is properly decoupled.

More geometric formulation: the Jacobi curve of a generic normal extremal is a product of curves in Lagrangian Grassmannians of smaller dimension)

Φ_* sends the Jacobi curve at λ of the corresponding extremal of g_1 to the Jacobi curve at $\Phi(\lambda)$ of the corresponding extremal g_2.

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THANK YOU VERY MUCH FOR YOUR ATTENTION!