Morse inequalities for eigenvalue branches of generic families of self-adjoint matrices

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Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

$$\mathrm{Sym}_n(\mathbb{K}) = \begin{cases} \text{the space of } n \times n \text{ symmetric matrices,} & \mathbb{K} = \mathbb{R} \\ \text{the space of } n \times n \text{ Hermitian matrices,} & \mathbb{K} = \mathbb{C}. \end{cases}$$

Given $A \in \mathrm{Sym}_n(\mathbb{K})$, let

$$\widehat{\lambda}_1(A) \leq \widehat{\lambda}_2(A) \leq \cdots \leq \widehat{\lambda}_n(A)$$

be the eigenvalues of A in the nondecreasing order.
$\widehat{\lambda}_k$ are Lipschitz continuous and it is nonsmooth at A if $\widehat{\lambda}_k(A)$ is a repeated eigenvalue of A.

Let M be a compact manifold.

$\mathcal{F} : M \to \mathrm{Sym}_n$ be a smooth map (a smooth family of self-adjoint matrices).

The kth branch of eigenvalues is the function $\lambda_k := \widehat{\lambda}_k \circ \mathcal{F}$.
A point $x_0 \in M$ is called Dirac's or diabolic point of λ_k if $\lambda_k(x_0)$ is a repeated eigenvalue of $\lambda_k(x_0)$.

In general, λ_k is Lipschitz but not smooth at x_0.

The goal: To construct the Morse theory of branches of eigenvalues, i.e. to inroduce the notion of critical points for diabolic points and to understand how the topology of sublevel sets is changing after the passages through the level set of a diabolic point.

Motivation:

The study of spectrum of Schrodinger operators with periodic potential via Bloch-Floquet theory:

- such operator is the direct integral of operators with discrete spectrum over the Brillion zone (the primitive cell of the reciprocal lattice of the lattice of periods of the potential)
- the spectrum has the band structure: the kth spectral band is the image of the k branch of eigenvalues of this family;
- diabolic points correspond to merging of spectral bands.

Review of classical Morse theory: basic terminology and notations

Let $f : M \to \mathbb{R}$ be a smooth function

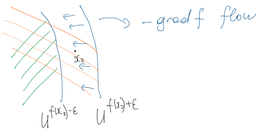
- A point $x_0 \in M$ is called critical if $df(x_0) = 0$ and regular if $df(x_0) \neq 0$;
- A critical point x_0 is called nondegenerate (or Morse critical) if the second derivative $d^2 f(x_0)$ at x_0 is nondegenerate ($\Leftrightarrow$ in local coordinates $(x_1, \ldots, x_n)$ the Hessian matrix $\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)$ is nondegenerate);
- The negative index of the second differential at x_0 is called the index of the critical point;
- The sublevel set of the value c is

$$M^c := \{x \in M : f(x) \leq c\}.$$

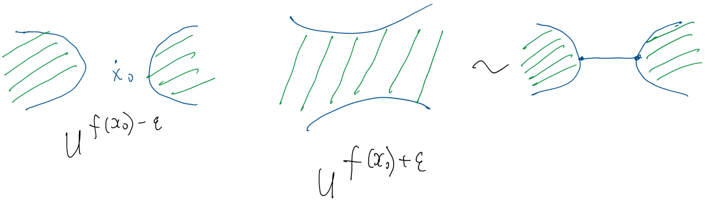
- The local sublevel set: If U is a neighborhood of x_0, $U^c := M^c \cap U$.

Review of classical Morse theory: topological change of sublevel set

1. If x_0 is a regular point, then for a sufficiently small neighborhood U of x_0 and $\varepsilon > 0$ the set $U^{f(x_0)-\varepsilon}$ is a deformation retract of $U^{f(x_0)+\varepsilon}$.



2. If x_0 is a Morse critical point of index μ, then $U^{f(x_0)+\varepsilon}$ is homotopy equivalent to $U^{f(x_0)-\varepsilon}$ with μ-dimensional cell attached.



Topologically regular and critical points of a continuous function

For a continuous (nonsmooth) function it is natural to use the properties of the previous slides as the definition of (topologically) regular / singular points of f (Fomenko-Fuks(1987), Goresky-McPerson (1988), Agrachev-Pallashke-Scholtes (1997)):

Definition

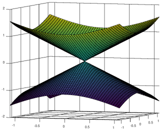
A point $x_0 \in M$ is called topologically regular point of f if there exists a neighborhood U of x_0 and $\varepsilon > 0$ such that $U^{f(x_0)-\varepsilon}$ is a deformation retract of $U^{f(x_0)+\varepsilon}$, and it is called topologically critical otherwise.

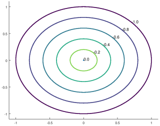
Two examples

Example

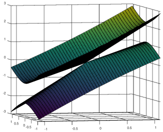
Consider the following two families:

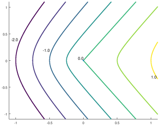
$$\mathcal{F}_1(x) = \begin{pmatrix} x_1 & x_2 \\ x_2 & -x_1 \end{pmatrix}, \quad \text{and} \quad \mathcal{F}_2(x) = \begin{pmatrix} x_1 & x_2 \\ x_2 & 2x_1 \end{pmatrix}, \quad x = (x_1, x_2) \in \mathbb{R}^2.$$





$(0,0)$ is topologically critical for $\mathcal{F}_1$.





$(0,0)$ is topologically regular for $\mathcal{F}_2$.

Question: *How to distinguish efficiently topologically regular and critical diabolic points?*

Sufficient condition for topological regularity.

Sym_ν is endowed with the natural Frobenius inner product $\langle X, Y\rangle = \mathrm{tr}(XY)$.

- Let
$$\mathrm{Sym}_\nu^+ = \{A \in \mathrm{Sym}_\nu : A \geq 0\},$$
the space of positive semidefinte self-adjoint matrices;
- Let
$$\mathrm{Sym}_\nu^{++} = \{A \in \mathrm{Sym}_\nu : A > 0\},$$
the space of positive definte self-adjoint matrices.

Ler x_0 be a diabolic point of λ_k such that $\mathcal{F}(x_0)$ has the multiplicity ν, E be the eigenspace of $\mathcal{F}(x_0)$ corresponding to the eigenvalue $\lambda_k(x_0)$, U be the $n \times \nu$ matrix with columns forming an orthonormal basis of E. Define $H_{x_0} : T_{x_0} \to \mathrm{Sym}_\nu$ by $H_{x_0}(v) = U^*(d\mathcal{F}(x_0)v)U$.

Theorem (G. Berkolaiko & I.Z.)

If $(\mathrm{Im}\, H_{x_0})^\perp \cap \mathrm{Sym}_\nu^+ = 0$ (equivalently, $\mathrm{Im}\, H_{x_0}$ contains a positively definite matrix), then x_0 is topologically regular (can be proved using Clarke subdifferential).

Conjecture on sufficient condition for topological criticality

Since the multiplicity $\nu(x)$ of the eigenvalue $\lambda_k(x)$ of the matrix $\mathcal{F}(x)$ is upper semicontinuous $\exists$ a neighborhood U of x_0 such that for all $x \in U$ $\nu(x) \leq \nu$ (here $\nu = \nu(x_0)$).

The set

$$S := \{x \in U : \nu(x) = \nu\}$$

is called the (local) constant degeneracy stratum attached to x_0.

Conjecture

If the following two conditions holds:

1. $(\mathrm{Im} H_{x_0})^{\perp} \cap \mathrm{Sym}_{\nu}^{++} \neq \emptyset$;
2. *the constant degeneracy stratum S is smooth and x_0 is the Morse critical point (in the classical sense) of the restriction $\lambda_k|_S$,*

then x_0 is a topologically critical point of λ_k.

We proved this conjecture in the case of transversal families (note that they form a generic set).

Transversal families

Let $Q^n_{k,\nu}$ be the subset of Sym_n consisting of the matrices whose eigenvalue λ_k has multiplicity ν.
$Q^n_{k,\nu}$ is a semialgebraic submanifold of Sym_n; $\mathrm{Discr}_n := \bigcup\limits_{1\le k\le n, \nu>1} Q^n_{k,\nu}$.
$\mathrm{codim} Q^n_{k,\nu} = s(\nu)$, where

$$s(\nu) := \dim\left(\mathrm{Sym}_\nu(\mathbb{K})\right) - 1 = \begin{cases} \frac{1}{2}\nu(\nu+1) - 1, & \mathbb{K} = \mathbb{R}, \\ \nu^2 - 1, & \mathbb{K} = \mathbb{C}. \end{cases}$$

The discriminant variety is $\mathrm{Discr}_n := \bigcup_{1\le k\le n, \nu>1} Q^n_{k,\nu}$.
The family $\mathcal{F}$ is called transversal if $\mathcal{F}$ is transversal to Discr_n, i.e. for every x such that $\lambda_k(x)$ is the eigenvalue of multiplicity ν

$$\mathrm{Im}\, d\mathcal{F}(x) + T_{\mathcal{F}(x)} Q^n_{k,\mu} = T_x \mathrm{Sym}_n.$$

Note that for x_0 with $\lambda_k(x_0)$ of multiplicity ν to appear in the transversal family we need $\dim M \ge s(\nu)$.

Sufficient conditions for diabolic critical points of transversal families

If $\mathcal{F}$ is a transversal family, then the constant degeneracy sets attached to every point are smooth submanifolds of codimension $s(\nu) \Rightarrow$ if x_0 is a smooth critical point of λ_k then
$\dim \mathrm{Im} H_{x_0} = \mathrm{codim} S_{x_0} = s(\nu) \Leftrightarrow \dim(\mathrm{Im}\, H_{x_0})^\perp = 1$.

Theorem (Gregory Berkolaiko & I. Z.)

If $\mathcal{F}$ is a transversal family and a point x_0 is such that the following two conditions hold:

1. $(\mathrm{Im}\, H_{x_0})^\perp = \mathrm{span}\, B$, *where B is positive definite;*
2. x_0 *is the Morse critical point (in the classical sense) of the restriction $\lambda_k|_S$, where S is the constant degeneracy stratum attached to x_0,*

then x_0 is a topologically critical point of λ_k.

Remark. Condition (1) of the previous theorem in fact implies that the Family $\mathcal{F}$ is transversal (to the discriminant set) at x_0.

Generalized Morse critical points and Morse families

A point x_0 satisfying conditions(1) and (2) of the previous theorem is called generalized Morse diabolic critical point and the family $\mathcal{F}$ for which all topologically critical points are either smooth Morse or generalized Morse diabolic critical points is called the generalized Morse family.

- The set of topologically critical points of a generalized Morse family is finite;
- The generalized Morse families are generic (i.e., open and dense) in the Whitney topology in $C^r(M, \mathrm{Sym}_n)$ for $2 \leq r \leq \infty$.

The Morse and Poincare polynomials

Assume that $f : M \to \mathbb{R}$ is a continuous function with a finite set $CP(f)$ of topological critical points and to any $x \in \mathrm{CP}(f)$ let $\beta_i(x)$ be the rank of the relaitive homology group $H_i(U^{f(x)+\varepsilon}, U^{f(x)-\varepsilon})$, i.e. the ith Betti number of $H_*(U^{f(x)+\varepsilon}, U^{f(x)-\varepsilon})$.
Let $P_{f,x}(t) := \sum_i \beta_i(x) t^i$.
For example, if x is the smooth Morse critical point of index $\mu(x)$, then $U^{f(x)+\varepsilon}/U^{f(x)-\varepsilon} = \mathbb{S}^{\mu(x)}$ and $P_{f,x}(t) = t^{\mu(x)}$.
The Morse polynomial P_f of the function f is given by

$$P_f := \sum_{x \in \mathrm{CP}(f)} P_{f,x}.$$

For example, if f is a smooth Morse function in the classical sense

$$P_f(t) := \sum_i (\#\text{ of critical points of } f \text{ of index } i) t^i.$$

Let $P_M(t)$ be the Poincare polynomial of M, $P_M(t) = \sum_i \beta_i(M) t^i$, where $\beta_i(M)$ is the Betti number of M.

The Morse inequality

For any continuous function f with finite number of topologically critical points there exists a polynomial $R(t)$ with nonnegative integer coefficients such that

$$P_f(t) - P_M(t) = (1+t)R(t). \qquad (1)$$

For example if f is a smooth Morse function in a classical sense, then (1) $\Leftrightarrow \forall j$

$$\sum_{i=0}^{j} (-1)^{j-i} (\#\text{ of critical points of } f \text{ of index } i) \geq \sum_{i=0}^{j} (-1)^{j-i} \beta_i(M)$$

and, in particular, $(\#\text{ of critical points of } f \text{ of index } j) \geq \beta_j(M)$.

Question If x_0 is a diabolic critical point for the branch of eigenvalues λ_k , what is its contribution P_{λ_k, x_0} to the Morse polynomial P_{λ_k}?

Main theorem

Let $\mathcal{F}$ be a generalized Morse family and let x_0 be the diabolic topological critical point of the λ_k such that:

- $\lambda_k(x_0)$ is an eigenvalue of multiplicity ν of $\mathcal{F}(x_0)$;
- i is the sequential number of λ_k among the eigenvalue branches equal to $\lambda_k(x_0)$ at x_0 counted from the top;
- μ is the (classical) index of x_0 as a smooth critical point of $\lambda_k|_S$, where S the constant degeneracy stratum attached to x_0.

Theorem (Gregory Berkolaiko & I.Z.)

For a sufficient small neighborhood U of x_0 and sufficiently small $\varepsilon > 0$

$$H_r(U^{\lambda_k(x_0)+\varepsilon}, U^{\lambda_k(x_0)-\varepsilon}) =$$

$$\begin{cases} H_{r-\mu-s(i)}\Big(\mathrm{Gr}_{\mathbb{R}}(i-1,\nu-1)\Big), & \mathbb{K}=\mathbb{R} \text{ and } i \text{ is odd,} \\ H_{r-\mu-s(i)}\Big(\mathrm{Gr}_{\mathbb{R}}\Big(i-1,\nu-1\Big)\Big),\widetilde{\mathbb{Z}}) & \mathbb{K}=\mathbb{R} \text{ and } i \text{ is even,} \\ H_{r-\mu-s(i)}\Big(\mathrm{Gr}_{\mathbb{C}}(i-1,\nu-1)\Big), & \mathbb{K}=\mathbb{C}. \end{cases}$$

Theorem (Gregory Berkolaiko & I.Z.)

For a sufficient small neighborhood U of x_0 and sufficiently small $\varepsilon > 0$

$$H_r(U^{\lambda_k(x_0)+\varepsilon}, U^{\lambda_k(x_0)-\varepsilon}) = \begin{cases} H_{r-\mu-s(i)}\Big(\mathrm{Gr}_{\mathbb{R}}(i-1, \nu-1)\Big), & \mathbb{K} = \mathbb{R} \textit{ and } i \textit{ is odd}, \\ H_{r-\mu-s(i)}\Big(\mathrm{Gr}_{\mathbb{R}}\Big(i-1, \nu-1\Big), \widetilde{\mathbb{Z}}\Big) & \mathbb{K} = \mathbb{R} \textit{ and } i \textit{ is even}, \\ H_{r-\mu-s(i)}\Big(\mathrm{Gr}_{\mathbb{C}}(i-1, \nu-1)\Big), & \mathbb{K} = \mathbb{C}. \end{cases}$$

- $\mathrm{Gr}_{\mathbb{K}}(j, l)$ is the Grassmannian of j-planes in $\mathbb{K}^l$;
- $s(i) = \dim \mathrm{Sym}_i - 1$;
- $H_*\Big(\mathrm{Gr}_{\mathbb{R}}\Big(i-1, \nu-1\Big), \widetilde{\mathbb{Z}}\Big)$ is the twisted homology, as described in the next slide.

Twisted homologies in more detail

The universal cover of $\mathrm{Gr}_{\mathbb{R}}(j,l)$ is a double cover and is isomorphic to the oriented Grassmanian $\widetilde{\mathrm{Gr}}_{\mathbb{R}}(j,l)$ consisting of the *oriented* j-dimensional subspaces in $\mathbb{R}^l$.

Let τ denote the orientation-reversing involution on $\widetilde{\mathrm{Gr}}_{\mathbb{R}}(j,l)$.

In the space of q-chains of $\widetilde{\mathrm{Gr}}_{\mathbb{R}}(j,l)$ over the ring $\mathbb{Z}$ we distinguish the subspace of chains which are skew-symmetric with respect to τ: $\tau(\alpha) = -\alpha$, where α is a chain.

The subspaces of skew-symmetric q-chains are invariant under the boundary operator and therefore define a complex.

The homology groups of this complex is called the twisted qth homology of $\mathrm{Gr}_{\mathbb{R}}(j,l)$ and is denoted by $H_q\big(\mathrm{Gr}_{\mathbb{R}}(j,l); \widetilde{\mathbb{Z}}\big)$.

Poincare polynomials of Grassmannians

Denote by $\binom{n}{k}_q$ the q-binomial coefficient,

$$\binom{n}{k}_q := \frac{\prod_{i=1}^{n}(1-q^i)}{\prod_{i=1}^{k}(1-q^i)\prod_{i=1}^{n-k}(1-q^i)}.$$

Then the Poincaré polynomial of the relative homology groups $H_r\left(U^{\lambda_k(x)+\varepsilon}, U^{\lambda_k(x)-\varepsilon}\right)$ is equal to

$$P_{\lambda_k}(t;x) := t^{\mu(x)+s(i)} \begin{cases} \binom{\lfloor(\nu-1)/2\rfloor}{(i-1)/2}_{t^4}, & \mathbb{K}=\mathbb{R} \text{ and } i \text{ is odd,} \\ 0, & \mathbb{K}=\mathbb{R},\ i \text{ is even, and } \nu \text{ is odd,} \\ t^{\nu-i}\binom{\nu/2-1}{i/2-1}_{t^4} & \mathbb{K}=\mathbb{R},\ i \text{ is even, and } \nu \text{ is even,} \\ \binom{\nu-1}{i-1}_{t^2} & \mathbb{K}=\mathbb{C}. \end{cases}$$

Contributions to the Morse polynomial in the real case for multiplicities not greater than 6 and isolated diabolic points

ν \ i	1	2	3	4	5	6
2	1	t^2				
3	1	0	t^5			
4	1	t^4	t^5	t^9		
5	1	0	t^5+t^9	0	t^{14}	
6	1	t^6	t^5+t^9	$t^{11}+t^{15}$	t^{14}	t^{20}

Table: Contributions to the Morse polynomial from a topologically critical point of $\lambda_k(x)$ in the real case ($\mathbb{K}=\mathbb{R}$). Only the "singular" (transversal) directions are considered; this corresponds to setting $\mu=0$ for $\mathbb{K}=\mathbb{R}$.

The generating function for the torsion part in the real case for multiplicities not greater than 6 and isolated diabolic points

ν \ i	1	2	3	4	5	6
2	0	0				
3	0	t^2	0			
4	0	t^2	t^6	0		
5	0	$t^2 + t^4$	$t^6 + t^7$	$t^9 + t^{11}$	0	
6	0	$t^2 + t^4$	$t^6 + t^7 + t^8 + t^{10}$	$t^9 + t^{11} + t^{12} + t^{13}$	$t^{15} + t^{17}$	0

Table: The generating functions of the torsion part (consisting of copies of $\mathbb{Z}_2$) of the relative homology groups $H_*\left(U^{\lambda_k(x_0)+\varepsilon}, U^{\lambda_k(x)-\varepsilon}\right)$ when the multiplicity of the considered eigenvalue is not greater than 6. Only the transversal directions are considered; this corresponds to setting $\mu = 0$ in the case $\mathbb{K} = \mathbb{R}$.

Corollary

Let x_0 be a non-degenerate topologically critical point of the eigenvalue λ_k of a smooth family $\mathcal{F} : M \to \mathrm{Sym}_n(\mathbb{K})$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Then x_0 is a local maximum (minimum) of λ_k if and only if the following two conditions hold simultaneously:

1. *the branch λ_k is the bottom (the top) branch among those coinciding with $\lambda_k(x)$ at x (equivalently, the relative index $i(x, k)$ is maximal (minimal) possible , i.e. is equal to $\nu(x)$ (equal to 1)).*
2. *the restriction of λ_k to the local constant degeneracy stratum attached to x has local maximum (minimum) at x.*

Sketch of the proof of the main theorem. Step 1: reduction to an isolated diabolic point

We use the Goresky-McPerson theory to split the study of sublevel sets of λ_k into the study of sublevel set of $\lambda_k|_S$ (the tangential data) and $\lambda_k|_N$ (the normal data), where S is the constant degeneracy stratum attached to x_0 and N is a submanifold through x_0 such that

$$T_{x_0}S \oplus T_{x_0}N = T_{x_0}M.$$

$\Downarrow$ (the Künneth theorem)

$$H_r(U^{\lambda_k(x_0)+\varepsilon}(\lambda_k)/U^{\lambda_k(x_0)-\varepsilon}(\lambda_k)) =$$
$$H_{r-\mu}(U^{\lambda_k(x_0)+\varepsilon}(\lambda_k|_N)/U^{\lambda_k(x_0)-\varepsilon}(\lambda_k|_N)),$$

so, we get the reduction to the case of transversal families with constant degeneracy strata being an isolated point, i.e. $N = M$

Sketch of the proof of the main theorem. Step 2: Homotopy equivalence to a Thom space over Grassmannians

Under the previous assumption ($N = M$) $U^{\lambda_k(x_0)+\varepsilon}(\lambda_k)/U^{\lambda_k(x_0)-\varepsilon}(\lambda_k)$ is homotopically equivalent to the suspension $\mathcal{S}R_{k,\nu}$ of the space

$$R_{k,\nu} = \{A \in \mathrm{Sym}_\nu^+ : \mathrm{tr} A = 1, \dim\ker A \geq k\}$$

Let $i = \nu - k + 1$.
Then (Agrachev, 2011) $\mathcal{S}R_{k,\nu}$ is homotopy equivbalent to the Thom space over the real bundle of rank $s(i)$ over the Grassmannian $\mathrm{Gr}_{\mathbb{K}}(i-1, \nu-1)$, which is orientable if i is odd and non-orientable if i is even $\Rightarrow$
Final step: We can use the Thom isomorphism theorem in the orientable case and some nonorientable analogue of it (or of the Poincare duality) for nonorientable case.

THANK YOU VERY MUCH FOR YOUR ATTENTION!