Projective and affine equivalence of sub-Riemannian metrics: integrability, generic rigidity, the Weyl type theorems, and separation of variables conjecture

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On projective and affine Equivalence of Riemannian metrics

Definition

Two Riemannian metrics g_1 *and* g_2 *on a manifold* M *are called* *projectively equivalent* *if they have the same geodesics , up to a reparametrization.*
They are called *affinely equivalent*, *if they have the same geodesics , up to an affine reparametrization.*

Two Riemannian metrics are affinely equivalent if and only if they have the same Levi-Civita connection.
Notation: $g_1 \overset{p}{\sim} g_2$ and $g_1 \overset{a}{\sim} g_2$, respectively.

Projective and affine rigidity and an example of a nonrigid metric

Obviously $g_1 \overset{a}{\sim} Cg_1$ for a positive constant C (we say that Cg_1 is constantly proportional to g_1).

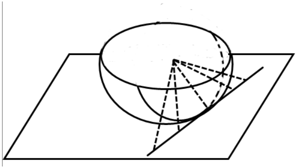
Definition

A metric on a connected manifold M is called projectively (affinely) rigid , if constantly proportional metrics are the only metrics which are projectively (affinely) equivalent to it.

Example of a projectively nonrigid metric

Example

The flat metric is not projectively rigid: If g_1 is the flat metric on a plane, g_2 is a standard metric on a hemisphere, and F is the stereographic projection from the center of the hemisphere to the plane (the gnomonic map projection) , then $(F^{-1})^* g_2 \sim g_1$ but they are not constantly proportional.



A Riemannian metric is (locally) projectively equivalent to the flat one if and only if it has constant curvature (Beltrami, 1865).

Transition operator and stabiliity.

All pairs of locally projectively equivalent Riemannian metrics with certain regularity assumption were described by Levi-Civita (1898), generalizing the previous result of Dini of 1869 on 2-dimensional case. These results exhibit certain separation of variables phenomenon.

Given two Riemannian metrics g_1 and g_2 let $S_q : T_qM \mapsto T_qM$ satisfy

$$g_{2_q}(v_1, v_2) = g_{1_q}(S_q v_1, v_2), \quad v_1, v_2 \in T_qM.$$

S_q is called the transition operator from the metrics g_1 to the metrics g_2 at the point q.

S_q is self-adjoint w.r.t. the Euclidean structure given by g_1.

A point $q_0 \in M$ is called stable w.r.t. the pair (g_1, g_2) if S_q has the same number of distinct eigenvalues in a neighborhood of q_0 (or, equivalently, the tuple of multiplicities of the eigenvalues is constant in a neighborhood of q_0).

Levi-Civita pairs of metrics

Definition

We say that two Riemannian metrics g_1 and g_2 constitute a Levi-Civita pair at a point q_0 if there exist positive integers $k_1, \dots k_m$, $\sum k_s = \dim M$, a local coordinate system $\bar{x} = (\bar{x}_1, \dots, \bar{x}_m)$, where $\bar{x}_s = (x_s^1, \dots, x_s^{k_s})$, and $\forall s, 1 \le s \le m$ a Riemannian metric b_s and a function β_s, both depending on variables $\bar{x}_s$ only and with β_s being constant if $k_s > 1$ and $\beta_s(q_0) \neq \beta_l(q_0)$ for all $s \neq l$, so that

$$g_1(\dot{\bar{x}}, \dot{\bar{x}}) = \sum_{s=1}^{m} \gamma_s(\bar{x}) b_s(\dot{\bar{x}}_s, \dot{\bar{x}}_s),$$

$$g_2(\dot{\bar{x}}, \dot{\bar{x}}) = \sum_{s=1}^{m} \lambda_s(\bar{x}) \gamma_s(\bar{x}) b_s(\dot{\bar{x}}_s, \dot{\bar{x}}_s).$$

where $\lambda_s(\bar{x}) = \beta_s(\bar{x}_s) \prod_{l=1}^{m} \beta_l(\bar{x}_l)$, $\gamma_s(\bar{x}) = \prod_{l \neq s} \left| \frac{1}{\beta_l(\bar{x}_l)} - \frac{1}{\beta_s(\bar{x}_s)} \right|$.

Levi-Civita and Eisenhart theorems

Theorem (Levi-Civita, 1898)

$g_1 \overset{p}{\sim} g_2$ *in a neighborhood of a stable point* $q_0 \in M$ *if and only if* g_1 *and* g_2 *form a Levi-Civita pair at* q_0*.*

Theorem (Eisenhart, 1923 for affine case)

$g_1 \overset{a}{\sim} g_2$ *in a neighborhood of a stable point* $q_0 \in M$ *if and only if* g_1 *and* g_2 *form a Levi-Civita pair at* q_0 *such that all functions* β_i *are constant.*

This theorem is also closely related to the classical De Rham Decomposition Theorem of a Riemannian manifolds in terms of the decomposition of the tangent bundle with respect to the holonomy group.

Case dim M=2 (Dini, 1869)

There exist local coordinates (x, y)

$$g_1 = (X(x) - Y(y))\,(dx^2 + dy^2)$$

$$g_2 = \left(\frac{1}{Y(y)} - \frac{1}{X(x)}\right)\left(\frac{dx^2}{X(x)} + \frac{dy^2}{Y(y)}\right).$$

Liouville surfaces

Existence of nontrivial quadratic integrals

Levi-Civita also showed that, in addition to the kinetic energy integral, the geodesic flow of g_1 admits $m-1$ integrals which are quadratic with respect to velocities (all these m integrals are in involution).
In particular, if $m>1$ it admits the following integral:

$$(\prod_{s=1}^{m} \lambda_s)^{-\frac{2}{m+1}} g_2(\dot{x}, \dot{x})$$

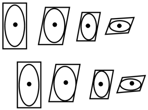
(Painlevè integral),a modern exposition in Matveev-Topalov, Geometria Dedicata, 2003. $\Rightarrow$
Generic Riemannian metrics are projectively rigid (as generic Riemannian metrics do not admit integrals which are quadratic (polynomial) in velocities, formal proof in Kruglikov-Matveev, Nonlinearity, 2016)).

Sub-Riemannian metrics

A rank ℓ distribution $D = \{D(q)\}_{q \in M}$ on a manifold M is a rank ℓ subbundle of the tangent bundle TM (a smooth field of ℓ-dimensional subspaces $D(q)$ of the tangent spaces $T_q M$).
D is called bracket-generating distribution if at any point iterated Lie brackets of vector fields tangent to D generate the whole tangent space.
Rashevsky-Chow Any two points of M can be connected by a curve tangent to a distribution.

A sub-Riemannian metric g is given on the distribution D, if an inner product g_q is chosen on each subspaces $D(q)$ smoothly in q.

Riemannian case: $D = TM$



Sub-Riemannian geodesics

Given a sub-Riemannian metric g, for any curve γ tangent to the distribution one can define the sub-Riemannian length by $\int g\big(\dot\gamma(t),\dot\gamma(t)\big)^{\frac{1}{2}}dt$.

Sub-Riemannian geodesics are the candidates for length-minimizers (via the Pontryagin Maximum Principle in Optimal Control).

Two types of geodesics:

- Abnormal -depend on the distribution D but not on the metric as unparametrized curves (no such geodesics in Riemannian case).
- Normal-projections to M of integral curves of the Hamiltonian system on T^*M corresponding to the Hamiltonian $h(p,q)=\frac{1}{2}||p|_{D(q)}||^2$ lying on the level set $h=\frac{1}{2}$ (in the Riemannian case these are exactly Riemannian geodesics).

Definition

Two sub-Riemannian metrics g_1 and g_2 on a distribution D are called projectively/affinely equivalent if they have the same normal geodesics, up to a reparametrization/an affine parametrization.

Distributions admitting product structure

Construction of pairs of projectively equivalent sub-Riemannian metrics by analogy with the metrics appearing in the Levi-Civita theorem:

Let $n = \dim M$. Fix positive integers $k_1, k_2, \dots k_m$ such that $n = k_1 + k_2 + \dots + k_m$. Let $\bar{x}_s = (x_s^1, \dots, x_s^{k_s})$ and $\bar{x} = (\bar{x}_1, \dots, \bar{x}_m)$ are standard coordinates in $\mathbb{R}^n = \mathbb{R}^{k_1} \times \mathbb{R}^{k_2} \times \dots \mathbb{R}^{k_m}$, where $\mathbb{R}^{k_s}$ has standard coordinates $\bar{x}_s$.

For any $1 \leq s \leq m$ let D_s be a bracket generating distribution in $\mathbb{R}^{k_s}$.

Consider the distribution D on $\mathbb{R}^n$ which is obtained by the product of distributions D_s.

Definition

We will say that a distribution admits a product structure, if it is locally equivalent to such distribution D with $m \geq 2$.

Generalized sub-Riemannian Levi-Civita pairs.

For every s, $1 \le s \le m$ choose a sub-Riemannian metric b_s on the distribution D_s of $\mathbb{R}^{k_s}$ and a function β_s depending on variables $\bar{x}_s$ only such that β_s is constant if $k_s > 1$ and $\beta_s(0) \neq \beta_l(0)$ for $s \neq l$.
Let

$$g_1(\dot{\bar{x}}, \dot{\bar{x}}) = \sum_{s=1}^{m} \gamma_s(\bar{x}) b_s(\dot{\bar{x}}_s, \dot{\bar{x}}_s),$$

$$g_2(\dot{\bar{x}}, \dot{\bar{x}}) = \sum_{s=1}^{m} \lambda_s(\bar{x}) \gamma_s(\bar{x}) b_s(\dot{\bar{x}}_s, \dot{\bar{x}}_s)$$

where the velocities $\dot{\bar{x}}$ belong to D, $\lambda_s(\bar{x}) = \beta_s(\bar{x}_s) \prod_{l=1}^{m} \beta_l(\bar{x}_l)$,
$\gamma_s(\bar{x}) = \prod_{l \neq s} \left| \frac{1}{\beta_l(\bar{x}_l)} - \frac{1}{\beta_s(\bar{x}_s)} \right|$.
Then $g_1 \overset{p}{\sim} g_2$ near the origin.

Also, the normal extremal flow of g_1 admits m integrals in involution as in Riemannian case.

The case of corank 1 distributions

Conjecture *The generalized Levi-Civita pairs are the only pairs of locally projectively equivalent sR metrics under certain regularity assumptions.*

Initial partial results in this direction (in the sequel we assume the stability of the transition operator):

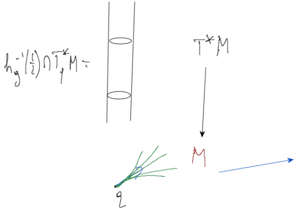
- sR metrics on contact distributions (I. Z., 2006). In this case it means that any sR metric is projectively rigid, because D does not admit product structure;
- sR metrics on quasi-contact distributions (I. Z. 2006). Generic sR metrics are projectively rigid;
- sR metric on corank 1 distributions with Cauchy characteristic being a sub-distribution (I. Z. and A. Castillo, 2014).

Reduction to the orbital equivalnece of Hamiltonioan flows

Projective/affine equivalence of g_1 and g_2 (with Hamiltonians h_1 and h_2) $\Leftrightarrow$ existence of the fiber-preserving preserving orbital diffeomorphism Φ between Hamiltonian flows on an open dense sets of the cotangent bundle, i.e.

$$\Phi_* \vec{h}_1 = a\vec{h}_2 \text{ on an open set of } T^*M.$$

(in affine case the function a must be constant along $\vec{h}_1$)



(dim M-rank D) *-parametric family of sR geodesics with the same initial velocity*

Normal sR geodesics are distinguished by higher jets and in different directions of D they might be distinguished by jets of different order.

Conformally and Weyl projectively rigidity

Definition

A sR metric g_1 is called conformally projectively rigid if $g_2 \stackrel{p}{\sim} g_1$ implies that g_2 is conformal to g_1.

Conformally projectively rigidity $\Rightarrow$ affine rigidity;

Definition

A sR metric g is said to be Weyl projectively rigid if any metric, which is simultaneously conformal to g and projectively equivalent to g is constantly proportional to g.

Theorem (Weyl 1921; Levi-Civita's Thm with spectral size 1)

For $\dim M > 1$ any Riemannian metric is Weyl projectively rigid.

Obviously, conformally $\&$ Weyl projectively rigidity $\Rightarrow$ projective rigidity.

Genericity of conformal projectively rigidity via non-integrability

Normal sub-Riemannian geodesics are projections to M of integral curves of the Hamiltonian system on T^*M corresponding to the sR Hamiltonian $h(p,q) = \frac{1}{2}||p|_{D(q)}||^2$ lying on the level set $h = \frac{1}{2}$. The integral curves of this Hamiltonian system are called normal extremals. The sub-Riemannian Hamiltonian is trivially an integral of the flow of normal extremals.

Theorem (Geom. Dedicata, 2019, arXiv:1801.04257v2)

If a sub-Riemannian metric is not conformally rigid, then the flow of its normal extremals admits a nontrivial integral quadratic in impulses (i.e. on the fibers of T^*M), namely the integral of Painlevè type.

Corollary

Generic sub-Riemannian metrics on a given distribution are conformally projectively rigid and therefore affinely rigid (and actually projectively rigid in real analytic category).

Tanaka symbol and nilpotent approximation of a distribution

D is called equiregular at q_0 if all D^j have constant dimension in a neighborhood of q_0.

Definition

- *The (Tanaka) symbol of an equiregular distribution D at a point q_0 is the graded nilpotent Lie algebra $D(q_0) \oplus D^2(q_0)/D(q_0) \oplus D^3(q_0)/D^2(q_0) \oplus \cdots$, generated by the first component (in this case it is called the fundamental nilpotent graded Lie algebra)*
- *The left-invariant distribution on the corresponding Lie group obtained by the left translation of $D(q_0)$ is called the nilpotent approximation of D at q_0 and is denote by $\widehat{D}_{q_0}$.*

Symbol and Nilpotent approximation of a sR structure

Definition

- *The symbol of an sR metric* g *is the pair consisting of the Tanaka symbol of* D *at* q_0 *and the Euclidean structure* $g(q_0)$ *on* $D(q_0)$*.*
- *The nilpotent approximation of sub-Riemannian metric* g *on an equiregular distribution* D *at a point* q_0 *is the left-invariant sR structure* $\hat{g}$ *on the Lie group of the Tanaka symbol of* D *at* q_0 *such that the Euclidean structure at the identity coincides with the Euclidean structure at* $D(q_0)$*.*

Direct product structure on the level of nilpotent approximation

Theorem (Geom. Dedicata, 2019, arXiv:1801.04257v2)

If g_1 and g_2 are two sub-Riemannian metric on an equiregular distribution D, which are locally projectively equivalent around a stable point q_0 and not conformal, then the nilpotent approximation $\hat{D}_{q_0}$ of D at q_0 admits a product structure and the corresponding nilpotent approximations $\hat{g}_1$ and $\hat{g}_2$ form a Levi-Civita pair with constant coefficients.

Corollary

- *If the Tanaka symbol of an equiregular distribution D at a stable point q_0 can not be represented as a direct sum of fundamental graded nilpotent Lie algebras, then any sub-Riemannian metric on D is affinely rigid and conformally projectively rigid.*
- *Any sub-Riemannian metric on a rank 2 bracket generating distribution is affinely rigid and conformally projectively rigid.*

Confirmation of the affine version of the conjecture for some decomposible Tanaka symboks

Conjecture (affine version) *The generalized Levi-Civita pairs with constant β's are the only pairs of locally affinely equivalent sR metrics under certain regularity assumptions $\Rightarrow$ an affinely nonrigid sub-Riemannian metric is a direct product of (at least) two sub-Riemannian metrics .*

Theorem (Zaifeng Lin , I. Z. , in preparation)

The affine version of the conjecture is true if each component in the decomposition of the Tanaka symbol of the underlying distribution is step 2 distribution (i.e. $D^2 = TM$) of corank not greater than 2.

Genericity of indecomposable fundamental graded Lie algebras

Let $\mathrm{GNLA}(m, n)$ be the set of all n-dimensional fundamental graded nilpotent Lie algebras generated such that the first component (which generates the Lie algebra) has dimension m.

Proposition

Except the following two cases:

1. $m = n - 1$ *with even* n,
2. $(m, n) = (4, 6)$,

a generic element of $\mathrm{GNLA}(m, n)$ *cannot be represented as a direct sum of two fundamental graded Lie algebras.*

Rigidity of SR structures on generic distribution

Theorem (Geom. Dedicata, 2019, arXiv:1801.04257v2)

Let m and n be two integers such that $2 \leq m < n$, and assume $(m, n) \neq (4, 6)$ and $m \neq n-1$ if n is even. Then, given an n-dimensional manifold M and a generic rank m distribution D on M, any sub-Riemannian metric on (M, D) conformally projectively rigid and therefore affinely rigid (and in the real analytic category even projectively rigid from the following sub-Riemannian Weyl results).

Theorem (Geom. Dedicata, 2021, arXiv:2001.08584)

Let m and n be two integers such that $2 \leq m < n$. On a generic real analytic rank m distribution D on a connected n-dimensional real analytic manifold M any sub-Riemannian metric is Weyl projectively rigid.

Decoupling/direct product on the level of Jacobi equations/Jacobi curves of extremals

Theorem (I.Z.)

If a sub-Riemannian metric is not affinely rigid then the Jacobi equation along generic normal extremal is properly decoupled.

More geometric formulation: the Jacobi curve of a generic normal extremal is a product of curves in Lagrangian Grassmannians of smaller dimension)

Question: *Does the fact that the Jacobi equation along generic normal extremal is properly decoupled implies that a sub-Riemannnian structure is a direct product of at least two sub-Riemannian metrics?*

Theorem (Zaifeng Lin and I.Z., in preparation)

The answer is "yes" for sub-Riemannian structures on even contact distributions and (4,6)-distributions.

References

1. I. Zelenko, *On geodesic equivalence of Riemannian metrics and sub-Riemannian metrics on distributions of corank 1*, J. Math. Sci. (N. Y.) 135 (2006), no. 4, 3168-3194.

2. F. Jean, S. Maslovskaya, and I. Zelenko, *Inverse Optimal Control Problem: the Sub-Riemannian Case*, Proceedings of IFAC (International Federation of Automatic Control) IFAC-papersOnLine , vol 50, 2017, 7 pages.

3. F. Jean, S. Maslovskaya, and I. Zelenko, *On projective and affine equivalence of sub-Riemannian metrics*, Geom. Dedicata 203 (2019), 279-319.

4. F. Jean, S. Maslovskaya, and I. Zelenko, *On Weyl's type theorems and genericity of projective rigidity in sub-Riemannian Geometry,* . Geom. Dedicata 213 (2021), 295–314

THANK YOU VERY MUCH FOR YOUR ATTENTION!