Workshop on "Geometry of vector distributions, differential equations, and variational problems"

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Abstracts of the talks

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If we generalize the notion of a manifold to that of a filtered manifold, the usual rôle of tangent space is played by the nilpotent graded Lie algebra which is defined at each point of the filtered manifold as its first order approximation. On the basis of this nilpotent approximation we have been studying various structures and objects on filtered manifolds to develop Nilpotent Geometry and Analysis.

In this talk we present a simple principle to associate systems of differential equations to a representation of a Lie algebra in the framework of nilpotent analysis.

Let $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ be a transitive graded Lie algebra, that is, a Lie algebra satisfying:

i) $[\mathfrak{g}_p, \mathfrak{g}_q] \subset \mathfrak{g}_{p+q}$

ii) $\dim \mathfrak{g}_- < \infty$, where $\mathfrak{g}_- = \bigoplus_{p<0} \mathfrak{g}_p$, the negative part of $\mathfrak{g}$

iii) (Transitivity) For $i \geq 0, x_i \in \mathfrak{g}_i$, if $[x_i, \mathfrak{g}_-] = 0$, then $x_i = 0$.

Let $V = \bigoplus_{q\in\mathbb{Z}} V_q$ be a graded vector space satisfying:

i) $\dim V_q < \infty$.

ii) There exists q_I such that $V_q = 0$ for $q \leq q_I$.

Let $\lambda : \mathfrak{g} \to \mathfrak{gl}(V)$ be a representation of $\mathfrak{g}$ on V such that

i) $\lambda(\mathfrak{g}_p)V_q \subset V_{p+q}$.

ii) There exists q_0 such that if $\lambda(\mathfrak{g}_-)x_q = 0$ for $q > q_0$ then $x_q = 0$.

We then consider the cohomology group $H(\mathfrak{g}_-, V) = \bigoplus_{p,r\in\mathbb{Z}} H^p_r(\mathfrak{g}_-, V)$ of the representation of $\mathfrak{g}_-$ on V, namely the cohomology group of the cochain complex:

$$\xrightarrow{\partial} \mathrm{Hom}(\wedge^{p-1}\mathfrak{g}_-, V)_r \xrightarrow{\partial} \mathrm{Hom}(\wedge^{p}\mathfrak{g}_-, V)_r \xrightarrow{\partial} \mathrm{Hom}(\wedge^{p+1}\mathfrak{g}_-, V)_r \xrightarrow{\partial}$$

where $\mathrm{Hom}(\wedge^p\mathfrak{g}_-, V)_r$ is the set of all homogeneous p-cochain ω of degree r, that is, $\omega(\mathfrak{g}_{a_1} \wedge \cdots \wedge \mathfrak{g}_{a_p}) \subset V_{a_1+\cdots+a_p+r}$ for any $a_1, \cdots, a_p < 0$.

Now our assertion may be roughly stated as follows:

Principle 1. *The first cohomology group $H^1(\mathfrak{g}_-, V) = \bigoplus H^1_r(\mathfrak{g}_-, V)$ represents a system of differential equations and $V = \bigoplus V_q$ represents its solution space.*

We will explain that it is in the framework of nilpotent analysis that the principle above is properly and well settled. We will also give several examples.

Key words : filtered manifold, weighted jet bundle, geometric structure and differential equation on a filtered manifold.