Workshop on "Geometry of vector distributions, differential equations, and variational problems"

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Abstracts of the talks

Aleksei Kushner (Astrakhan State University and Control Sciences Institute of the Russian Academy of Sciences, Moscow, Russia) *Invariants of almost-product structures and geometry of Monge-Ampere and Jacobi equations*

Let N be a smooth manifold and let $\mathcal{P} = (P_1, \dots, P_r)$ be an ordered set of real (or complex) distributions on N, i.e.

$$P_i : N \ni a \mapsto P_i\,(a) \subset T_a N \quad (\text{or } T_a N^{\mathbb{C}}).$$

The set $\mathcal{P}$ is called an *almost product structure* or a *$\mathcal{P}$-structure* [1] on N if at each point $a \in N$ the tangent space $T_a N$ (for real distributions) or its complexification $T_a N^{\mathbb{C}}$ (for complex ones) splits in the direct sum of the subspaces $P_1\,(a), \dots, P_r\,(a)$, i.e. $T_a N = \oplus_{i=1}^{r} P_i\,(a)$ or $T_a N^{\mathbb{C}} = \oplus_{i=1}^{r} P_i\,(a)$.

We get the following decomposition of the de Rham complex: the $C^\infty\,(N)$-modules of differential s-forms $\Omega^s\,(N)$ split in the direct sum

$$\Omega^s\,(N) = \underset{|\mathbf{k}|=s}{\oplus} \Omega^{\mathbf{k}}\,(N)\,, \tag{0.1}$$

where $\mathbf{k}$ is a multiindex, $\mathbf{k} = (k_1, \dots, k_r)$, $k_i \in \{0, 1, \dots, n_i\}$, $n_i = dim P_i$, $|\mathbf{k}| = k_1 + \cdots + k_r$,

$$\Omega^{\mathbf{k}}\,(N) = \otimes_{i=1}^{r} \Omega$$

and

$$\Omega^{k_i}(P_i) = \left\{ \alpha \in \Omega^{k_i}\,(N) \middle| X \rfloor \alpha = 0 \ \forall X \in D(P_1) \oplus \cdots \oplus \widehat{D\,(P_i)} \oplus \cdots \oplus D\,(P_r) \right\}.$$

In case of complex almost product structures we have to consider the complexification $\Omega^s\,(N)^{\mathbb{C}}$ of the module $\Omega^s\,(N)$.

The de Rham differential d splits in the following direct sum:

$$d = \underset{|\mathbf{t}|=1}{\oplus} d_{\mathbf{t}},$$

where $t_j \in I_j = \{\, z \in \mathbb{Z} |\, |z| \le \dim P_j \}$ and

$$d_{\mathbf{t}} : \Omega^{\mathbf{k}}(N) \to \Omega^{\mathbf{k}+\sigma}(N).$$

Theorem 1. *If one of the component t_i of a multi-index $\mathbf{t}$ is negative, then operator $d_{\mathbf{t}}$ is a $C^\infty(N)$-homomorphism.*

In the other words, if one of the components t_i of the multiindex $\mathbf{t}$ is negative, then operator $d_{\mathbf{t}}$ is tensor which acts from $\Omega^{\mathbf{k}}(N)$ to $\Omega^{\mathbf{k}+\mathbf{t}}(N)$. Such tensors are invariant with respect to diffeomorphisms.

Any (hyperbolic or elliptic) classical Monge-Ampère equation on a two-dimensiomnal manifold M

$$Av_{xx} + 2Bv_{xy} + Cv_{yy} + D\left(v_{xx}v_{yy} - v_{xy}^2\right) + E = 0. \tag{0.2}$$

is an almost product structure (real or complex) on the manifold of 1-jets $J^1(M)$ (see [1]).

Jacobi equations [1]

$$\begin{cases} A_1 + B_1\dfrac{\partial v_1}{\partial x} - C_1\dfrac{\partial v_1}{\partial y} - D_1\dfrac{\partial v_2}{\partial y} + E_1\dfrac{\partial v_2}{\partial x} + F_1 \det J_v = 0, \\ A_2 + B_2\dfrac{\partial v_1}{\partial x} - C_2\dfrac{\partial v_1}{\partial y} - D_2\dfrac{\partial v_2}{\partial y} + E_2\dfrac{\partial v_2}{\partial x} + F_2 \det J_v = 0, \end{cases}$$

can be regarded as almost product structures on a 4-dimensional manifold. Here

$$\det J_v = \begin{vmatrix} \dfrac{\partial v_1}{\partial x} & \dfrac{\partial v_1}{\partial y} \\ \dfrac{\partial v_2}{\partial y} & \dfrac{\partial v_2}{\partial x} \end{vmatrix}$$

is a Jacobian and $A_i, B_i, C_i, D_i, E_i, F_i$ $(i = 1, 2)$ are some functions on x, y, v.

We applly constructed tensors to the problem of classification of Monge-Ampère and Jacobi equations solve the problem linearization of ones.

[1] Kushner A., Lychagin V., Ruvtsov V., *Contact geometry of non-linear differential equations*, Cambbridge Universiry Press, Cambridge, UK, (2006) (in press).