On geometry of affine control systems with one input

Boris Doubrov and Igor Zelenko

Dedicated to Andrei Agrachev, our teacher and mentor,
on the occasion of his 60th birthday.

Abstract We demonstrate how the novel approach to the local geometry of structures of nonholonomic nature, originated by Andrei Agrachev, works for rank 2 distributions of maximal class in $\mathbb{R}^n$ with additional structures such as affine control systems with one input spanning these distributions, sub-(pseudo)Riemannian structures etc. In contrast to the case of an arbitrary rank 2 distribution without additional structures, in the considered cases each abnormal extremal (of the underlying rank 2 distribution) possesses a distinguished parametrization. This fact allows one to construct the canonical frame on a $(2n-3)$-dimensional for arbitrary $n \geq 5$. The moduli spaces of the most symmetric models are described as well.

Key words: Abnormal extremals, affine control systems, micro-local state-feedback equivalence, self-dual curves in projective space, Wilczynski invariants, symplectic invariants, canonical frames, maximally symmetric models.

1 Introduction

About seventeen years ago Andrei Agrachev proposed the idea to study the local geometry of control systems and geometric structures on manifolds by studying the flow of extremals of optimal control problems naturally associated with these ob-

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jects [1, 2, 3]. Originally he considered situations when one can assign a curve of Lagrangian subspaces of a linear symplectic space or, in other words, a curve in a Lagrangian Grassmannian to an extremal of these optimal control problems. This curve was called the Jacobi curve of this extremal, because it contains all information about the solutions of the Jacobi equations along it. Agrachev's constructions of Jacobi curves worked in particular for normal extremals of sub-Riemannian structures and abnormal extremals of rank 2 distributions. Similar idea can be used for abnormal extremals of distribution of any rank, resulting in more general curves of coisotropic subspaces in a linear symplectic space [11, 15].

The key point is that the differential geometry of the original structure can be studied via differential geometry of such curves with respect to the action of the linear symplectic group. The latter problem is simpler in many respects than the original one. In particular, any symplectic invariants of the Jacobi curves produces the invariant of the original structure.

This idea proved to be very prolific. For the geometry of distributions, first it led to a new geometric-control interpretation of the classical Cartan invariant of rank 2 distributions on a five dimensional manifold, relating it to the classical Wilczynski invariants of curves in projective spaces [24, 23, 4]. It also gave a new effective method of the calculation of the Cartan tensor and the generalization of the latter invariant to rank 2 distributions on manifolds of arbitrary dimensions. These new invariants are obtained from the Wilczynski invariants of curves in projective spaces, induced from the Jacobi curves by a series of osculations together with the operation of taking skew symmetric complements. They are called the *generalized Wilczynski invariants of rank 2 distributions* (see section 5 for details).

Later on, we used this approach for the construction of the canonical frames for rank 2 distributions on manifolds of arbitrary dimension [9, 10], and, in combination with algebraic prolongation techniques in a spirit of N. Tanaka, for the construction of the canonical frames for distributions of rank 3 [11] and recently of arbitrary rank [15, 16] under very mild genericity assumptions called maximality of class. Remarkably, these constructions are independent of the nilpotent approximation (the Tanaka symbol) of a distribution at a point and even independent of its small growth vector. This extends significantly the scope of distributions for which the canonical frames can be constructed explicitly and in an unified way compared to the Tanaka approach ([20, 18, 6, 26]).

Perhaps the case of rank 2 distributions of so-called maximal class in $\mathbb{R}^n$ with $n > 5$ provides the most illustrative example of the effectiveness of this approach, because the construction of the canonical frame in this case needs nothing more than some simple facts from the classical theory of curves in projective spaces such as the existence of the canonical projective structure on such curves, i.e. a special set of parametrizations defined up to a Möbius transformation (see section 5 below). The canonical frame for such distributions is constructed in a unified way on a bundle of dimension $2n-1$ and this dimension cannot be reduced, because there exists the unique, up to a local equivalence, rank 2 distribution of maximal class in $\mathbb{R}^n$ with the pseudo-group of local symmetries of dimension equal to $2n-1$. For this most symmetric rank 2 distribution of maximal class all generalized Wilczynski invariants are identically zero.

However, under some additional assumptions, the canonical parametrization, up to a shift, on abnormal extremals can be distinguished instead of the canonical projective structure and one would expect that the canonical frame can be constructed on a bundle of smaller dimension.

What are these additional assumptions? One possibility is to consider rank 2 distributions of maximal class such that at least one of its generalized Wilczynski invariant does not vanish. Due to the size limits for the paper we postpone the treatment of this case to another paper (see also preprint [13]).

Another possibility is to consider a rank 2 distribution D with the additional structures defining a control system with one input satisfying certain regularity assumptions. A *control system with one input on a distribution D in the manifold M* is given by choosing a one-dimensional submanifold $\mathscr{V}_q$ on each fiber $D(q)$ of the distribution D for any point $q \in M$ (smoothly depending on q). The set $\mathscr{V}_q \subset D(q)$ is called the *set of admissible velocities of the control system at q.*

Let us introduce several natural notions of equivalence of control systems. We say that two control systems given by one-dimensional submanifolds $\mathscr{V}_q$ and $\widetilde{\mathscr{V}_q}$ on each fiber $D(q)$ are *(state-feedback) equivalent* if there exists a diffeomorphism F of M such that

$$F_*(\mathscr{V}_q) = \widetilde{\mathscr{V}}_{F(q)} \tag{1.1}$$

for any $q \in M$. These control systems are called *locally equivalent* at the points q_0 and $\tilde{q}_0$ of M, respectively, if there exists neighborhoods U and $\widetilde{U}$ of q_0 and $\tilde{q}_0$ in M, respectively, and a diffeomorphism $F\colon\ : U \to \widetilde{U}$ such that (1.1) holds for any $q \in U$. Finally, these control systems are called *micro-locally equivalent* at (q_0, v_0) and $(\tilde{q}_0, \tilde{v}_0)$, where the points q_0 and $\tilde{q}_0$ belong to M, $v \in V_q$, and $\tilde{v} \in V_q$, if there exist neighborhoods $\mathfrak{U}$ and $\widetilde{\mathfrak{U}}$ of (q_0, v_0) and $(\tilde{q}_0, \tilde{v}_0)$ in the set $\mathfrak{V} = \{(q, v) : q \in M, v \in V_q\}$ and a diffeomorphism $F\colon\ : \mathrm{pr}(\mathfrak{U}) \to \mathrm{pr}(\widetilde{\mathfrak{U}})$, where $\mathrm{pr}\colon \mathfrak{V} \to M$ is the canonical projection, such that $F_* v \in \mathscr{V}_{F(q)} \cap \widetilde{\mathfrak{U}}$ for any $(q, v) \in \mathfrak{U}$. From these notions of equivalence one can define the group of symmetries and pseudo-groups of local and micro-local symmetries of a control system accordingly. In the paper we mainly work with the micro-local equivalence but if one restricts himself to affine control systems only, then in all formulations the micro-local equivalence can be replaces by the local one.

Definition 1. Consider a control system with one input on a distribution D with the set of admissible velocities $\mathscr{V}_q$ at a point q. A line in $D(q)$ (through the origin) intersecting the set $\mathscr{V}_q \backslash \{\text{the origin of } D(q)\}$ in a finite number of points is called a *regular line* of the control system at the point q.

Definition 2. We say that a control system with one input on a rank 2 distribution D is *regular* if for any point q the sets of regular lines is a nonempty open subset of the projectivization $\mathbb{P}D(q)$.

An important particular class of examples of such control systems is when $\mathscr{V}_q$ is an affine line. In this case we get an *affine control system with one input and with* *a non-zero drift*. Another examples are sub-(pseudo)Riemannian structures, when the curves are ± 1-level sets of non-degenerate quadrics. For affine control systems with a non-zero drift and sub-Riemannian structures all lines in $D(q)$ are regular, while for sub-pseudo-Riemannian case all lines except the asymptotic lines of the quadrics are regular.

The goal of this paper is to demonstrate the approach, originated by Andrei Agrachev, in this simplified but still important situation of regular control system with one input on rank 2 distributions of maximal class. We show that *in this situations the canonical frame can be constructed in a unified way on a bundle of dimension* $2n-3$ *for all* $n \geq 5$ (Theorem 3, section 7). We also describe all models with the pseudo-group of micro-local symmetries of dimension $2n-3$. i.e. the most symmetric ones, among the considered class of objects (Theorem 1 below and its reformulation in Theorem , section 9).

The most symmetric models depend on continuous parameters. Let us describe these models. Given a tuple of $n-3$ constants $(r_1,\dots,r_{n-3})$ let $A_{(r_1,\dots,r_{n-3})}$ be the following affine control system in R^n taken with coordinates $(x,y_0,\dots,y_{n-3},z)$:

$$\dot{q} = X_1\big(q\big) + uX_2(q), \tag{1.2}$$

where

$$\begin{aligned} X_1 = {} & \frac{\partial}{\partial x} + y_1\frac{\partial}{\partial y_0} + \dots + y_{n-3}\frac{\partial}{\partial y_{n-4}} + \\ & \big(y_{n-3}^2 + r_1 y_{n-4}^2 + r_2 y_{n-5}^2 + \dots r_{n-3} y_0^2\big)\frac{\partial}{\partial z}, \end{aligned} \tag{1.3}$$

$$X_2 = \frac{\partial}{\partial y_{n-3}}. \tag{1.4}$$

and denote by $D_{(r_1,\dots,r_{n-3})}$ the corresponding rank 2 distribution generated by the vector fields X_1 and X_2 as in (1.3)-(1.4). Note that, as shown in [9, 10], the most symmetric rank 2 distribution in R^n of maximal class with $n \geq 5$ is locally equivalent to $D_{(0,\dots,0)}$. In the case of regular control systems we prove the following

Theorem 1. *A regular control systems with one input on a rank 2 distribution of maximal class in* $\mathbb{R}^n$ *with* $n \geq 5$ *has the pseudo-group of micro-local symmetries of dimension not greater than* $2n-3$. *If this dimension is equal to* $2n-3$, *then the control system is micro-locally equivalent to the system* $A_{(r_1,\dots,r_{n-3})}$ *for some constants* $r_i \in \mathbb{R}$, $1 \leq i \leq n-3$. *The affine control systems* $A_{(r_1,\dots,r_{n-3})}$ *corresponding to the different tuples* $(r_1,\dots,r_{n-3})$ *are not equivalent.*

Rephrasing the last sentence of the Theorem 1, the map $(r_1,\dots,r_{n-3}) \mapsto A_{(r_1,\dots,r_{n-3})}$ identifies the space $\mathscr{A}_n$ of the most symmetric, up to a micro-local equivalence, regular control systems on rank 2 distributions of maximal class in $\mathbb{R}^n$ with $\mathbb{R}^{n-3}$.

Remark 1. (see [13] for more detail) Note that the underlying distributions $D_{(r_1,\dots,r_{n-3})}$ might be equivalent for different tuples $(r_1,\dots,r_{n-3})$. Among all distributions of the type $D_{(r_1,\dots,r_{n-3})}$ there is a one-parametric family of distributions which are locally equivalent to $D_{(0,\dots,0)}$. To describe this family we say that a tuple of m numbers $(r_1,\dots,r_m)$ is called exceptional if the roots of the polynomial

$$\lambda^{2m}+\sum_{i=1}^{m}(-1)^i r_i\lambda^{2(m-i)} \tag{1.5}$$

constitute an arithmetic progression (with the zero sum in this case). Equivalently, $(r_1,\dots,r_m)$ is exceptional if $r_i=\alpha_{m,i}\left(\frac{r_1}{\alpha_{m,1}}\right)^i$, $1\le i\le m$, where the constants $\alpha_{m,i}$, $1\le i\le m$, satisfy the following identity

$$x^{2m}+\sum_{i=1}^{m}(-1)^i\alpha_{m,i}x^{2(m-i)}=\prod_{i=1}^{m}\bigl(x^2-(2i-1)^2\bigr). \tag{1.6}$$

The distribution $D_{(r_1,\dots,r_{n-3})}$ is locally equivalent to the distribution $D_{(0,\dots,0)}$ (or, equivalently, has the algebra of infinitesimal symmetries of the maximal possible dimension among all rank 2 distributions of maximal class in $\mathbb{R}^n$) if and only if the tuple $(r_1,\dots,r_{n-3})$ is exceptional. The distribution $D_{(\tilde r_1,\dots,\tilde r_{n-3})}$ is locally equivalent to the distribution $D_{(r_1,\dots,r_{n-3})}$, where the tuple $(r_1,\dots,r_{n-3})$ is not exceptional, if and only if

$$\text{there exists } c\neq 0 \text{ such that } \tilde r_i=c^{2i}r_i,\quad 1\le i\le n-3. \quad \square$$

Finally note that affine control systems with one input were considered also in [5], but the genericity assumptions imposed there are much stronger than our genericity assumptions here.

The paper is organized as follows. The main results are given in sections 7 and 9 (Theorem 3 and Theorem 4, which are reformulations of Theorem 1 above). Sections 2-6 are preparatory for section 7, section 8 is preparatory for section 9. In sections 2-5 we list all necessary facts about abnormal extremals of rank 2 distributions, their Jacobi curves and describe the canpnical projective structure on a unparametrized curve in projective spaces. The details can be found in [9, 23, 22]. In section 6 we summarize the main results of [9, 10] about canonical frames for rank 2 distributions of maximal class in order to compare them with the analogous results of sections 7 and 9. In section 8 we list all necessary facts about the invariants of parametrized self-dual curves in projective spaces.

2 Abnormal extremals of rank 2 distributions

Let D be a rank 2 distribution on a manifold M. A smooth section of a vector bundle D is called a *horizontal vector field of D*. Taking iterative brackets of horizontal vector fields of D, we obtain the natural filtration $\{\dim D^j(q)\}_{j\in\mathbb{N}}$ on each tangent space T_qM. Here D^j is the j-th power of the distribution D, i.e., $D^j=D^{j-1}+[D,D^{j-1}]$,

$D^1 = D$, or , equivalently, $D^j(q)$ is a linear span of all Lie brackets of the length not greater than j of horizontal vector fields of D evaluated at q.

Assume that $\dim D^2(q) = 3$ and $\dim D^3(q) > 3$ for any $q \in M$. Denote by $(D^j)^\perp \subset T^*M$ the annihilator of the jth power D^j, namely

$$(D^j)^\perp = \{(p,q) \in T^*M : \ p \cdot v = 0 \ \forall v \in D^j(q)\}.$$

Recall that abnormal extremals of D are by definition the Pontryagin extremals with the vanishing Lagrange multiplier near the functional for any extremal problem with constrains, given by the distribution D. They depend only on the distribution D and not on a functional.

It is easy to show (see, for example, [22, 10]) that for rank 2 distributions all abnormal extremals lie in $(D^2)^\perp$ and that through any point of the codimension 3 submanifold $(D^2)^\perp \backslash (D^3)^\perp$ of T^*M passes exactly one abnormal extremal or, in other words, $(D^2)^\perp \backslash (D^3)^\perp$ is foliated by the characteristic 1-foliation of abnormal extremals. To describe this foliation let $\pi : T^*M \mapsto M$ be the canonical projection. For any $\lambda \in T^*M$, $\lambda = (p,q)$, $q \in M$, $p \in T^*_qM$, let $\mathfrak{s}(\lambda)(\cdot) = p(\pi_* \cdot)$ be the canonical Liouville form and $\sigma = d\mathfrak{s}$ be the standard symplectic structure on T^*M. Since the submanifold $(D^2)^\perp$ has odd codimension in T^*M, the kernels of the restriction $\sigma|_{(D^2)^\perp}$ of σ on $(D^2)^\perp$ are not trivial. At the the points of $(D^2)^\perp \backslash (D^3)^\perp$ these kernels are one-dimensional. They form the *characteristic line distribution* in $(D^2)^\perp \backslash (D^3)^\perp$, which will be denoted by $\mathscr{C}$. The line distribution $\mathscr{C}$ defines the desired *characteristic 1-foliation* on $(D^2)^\perp \backslash (D^3)^\perp$ and the leaf of this foliation through a point is exactly the abnormal extremal passing through this point. From now on we shall work with abnormal extremals which are integral curves of the characteristic distribution $\mathscr{C}$.

The characteristic line distribution $\mathscr{C}$ can be easily described in terms of a local basis of the distribution D, i.e. two horizontal vector fields X_1 and X_2 such that $D(q) = \operatorname{span}\{X_1(q), X_2(q)\}$ for all q from some open set of M. Denote by

$$X_3 = [X_1, X_2],\ X_4 = \big[X_1, [X_1, X_2]\big],\ X_5 = \big[X_2, [X_1, X_2]\big]. \tag{2.1}$$

Let us introduce the "quasi-impulses" $u_i : T^*M \mapsto \mathbb{R}$, $1 \le i \le 5$,

$$u_i(\lambda) = p \cdot X_i(q),\ \lambda = (p,q),\ q \in M,\ p \in T^*_qM. \tag{2.2}$$

Then by the definition

$$(D^2)^\perp = \{\lambda \in T^*M : u_1(\lambda) = u_2(\lambda) = u_3(\lambda) = 0\}. \tag{2.3}$$

As usual, for a given function $h : T^*M \mapsto \mathbb{R}$ denote by $\overrightarrow{h}$ the corresponding Hamiltonian vector field defined by the relation $i_{\overrightarrow{h}}\sigma = -d\,h$. Then by the direct computations (see, for example, [10]) the characteristic line distribution $\mathscr{C}$ satisfies

$$\mathscr{C} = \operatorname{span}\{\mathrm{u}_4 \overrightarrow{u}_2 - \mathrm{u}_5 \overrightarrow{u}_1\}. \tag{2.4}$$

3 Jacobi curves of abnormal extremals

Now we are ready to define the Jacobi curve of an abnormal extremal of D. For this first lift the distribution D to $(D^2)^\perp$, namely considered the distribution $\mathscr{J}$ on $(D^2)^\perp$ such that

$$\mathscr{J}(\lambda) = \{v \in T_\lambda (D^2)^\perp : d\pi(v) \in D(\pi(\lambda))\}. \tag{3.1}$$

Note that $\dim \mathscr{J} = n-1$ and $\mathscr{C} \subset \mathscr{J}$ by (2.4) . The distribution $\mathscr{J}$ is called the *lift of the distribution D to $(D^2)^\perp \backslash (D^3)^\perp$*.

Given a segment γ of an abnormal extremal (i.e. of a leaf of the 1-characteristic foliation) of D, take a sufficiently small neighborhood O_γ of γ in $(D^2)^\perp$ such that the quotient $N = O_\gamma/(\textit{the characteristic one-foliation})$ is a well defined smooth manifold. The quotient manifold N is a symplectic manifold endowed with the symplectic structure $\bar{\sigma}$ induced by $\sigma|_{(D^2)^\perp}$. Let

$$\phi : O_\gamma \to N \tag{3.2}$$

be the canonical projection on the factor. Define the following curves of subspaces in $T_\gamma N$:

$$\lambda \mapsto \phi_*\big(\mathscr{J}(\lambda)\big), \quad \forall \lambda \in \gamma. \tag{3.3}$$

Informally speaking, these curves describe the dynamics of the distribution $\mathscr{J}$ w.r.t. the characteristic 1-foliation along the abnormal extremal γ.

Note that there exists a straight line, which is common to all subspaces appearing in (3.3) for any $\lambda \in \gamma$. So, it is more convenient to get rid of it by a factorization. Indeed, let e be the Euler field on T^*M, i.e., the infinitesimal generator of homotheties on the fibers of T^*M. Since a transformation of T^*M, which is a homothety on each fiber with the same homothety coefficient, sends abnormal extremals to abnormal extremals, we see that the vector $\bar{e} = \phi_* e(\lambda)$ is the same for any $\lambda \in \gamma$ and lies in any subspace appearing in (3.3). Let

$$J_\gamma(\lambda) = \phi_*\big(\mathscr{J}(\lambda)\big)/\{\mathbb{R}\bar{e}\}, \quad \forall \lambda \in \gamma \tag{3.4}$$

The (unparametrized) curve $\lambda \mapsto J_\gamma(\lambda)$, $\lambda \in \gamma$ is called the *Jacobi curve of the abnormal extremal* γ. It is clear that all subspaces appearing in (3.4) belong to the space

$$W_\gamma = \{v \in T_\gamma N : \bar{\sigma}(v, \bar{e}) = 0\}/\{\mathbb{R}\bar{e}\}. \tag{3.5}$$

and that

$$\dim J_\gamma(\lambda) = n-3. \tag{3.6}$$

The space W_γ is endowed with the natural symplectic structure $\tilde{\sigma}_\gamma$ induced by $\bar{\sigma}$. Also $\dim W_\gamma = 2(n-3)$.

Given a subspace L of W_γ denote by $L^\angle$ the skew-orthogonal complement of L with respect to the symplectic form $\tilde{\sigma}_\gamma$, $L^\angle = \{v \in W_\gamma, \sigma_\gamma(v, \ell) = 0 \quad \forall \ell \in L\}$. Recall that the subspace L is called *isotropic* if $L \subseteq L^\angle$, *coisotropic* if $L^\angle \subseteq L$, and *La-*

grangian, if $L = L^{\angle}$. Directly from the definition, the dimension of an isotropic subspace does not exceed $\frac{1}{2}\dim W_\gamma$, and a Lagrangian subspace is an isotropic subspace of the maximal possible dimension $\frac{1}{2}\dim W_\gamma$. The set of all Lagrangian subspaces of W_γ is called the *Lagrangian Grassmannian of* W_γ.

It is easy to see ([10, 23]) that the Jacobi curve of an abnormal extremal consists of Lagrangian subspaces, i.e. it is a curve in the Lagrangian Grassmannian of W_γ. In the case $n \geq 5$ (equivalently, $\dim W_\gamma \geq 4$) curves in the Lagrangian Grassmannian of W_γ have a nontrivial geometry with respect to the action of the linear symplectic group and any symplectic invariant of Jacobi curves of abnormal extremals produces an invariant of the original distribution D.

4 Reduction to geometry of curves in projective spaces

In the earlier works [3, 23] invariants of Jacobi curves were constructed using the notion of the cross-ratio of four points in Lagrangian Grassmannians analogous to the classical cross-ratio of four point in a projective line. Later, we developed a different method, leading to the construction of canonical bundles of moving frames and invariants for quite general curves in Grassmannians and flag varieties [12, 14]. The geometry of Jacobi curves J_γ in the case of rank 2 distributions can be reduced to the geometry of the so-called self-dual curves in the projective space $\mathbb{P}W_\gamma$.

For this first one can produce a curve of flags of isotropic/coisotropic subspaces of W_γ by a series of osculations together with the operation of taking skew symmetric complements. For this, denote by $C(J_\gamma)$ the *tautological bundle* over J_γ: the fiber of $C(J_\gamma)$ over the point $J_\gamma(\lambda)$ is the linear space $J_\gamma(\lambda)$. Let $\Gamma(J_\gamma)$ be the space of all smooth sections of $C(J_\gamma)$. If $\psi : (-\varepsilon, \varepsilon) \mapsto \gamma$ is a parametrization of γ such that $\psi(0) = \lambda$, then for any $i \geq 0$ define

$$J_\gamma^{(i)}(\lambda) := \operatorname{span}\{\tfrac{d^j}{d\tau^j}\ell\big(\psi(t)\big)\big|_{t=0} : \ell \in \Gamma(J_\gamma), 0 \leq j \leq i\} \tag{4.1}$$

$$J_\gamma^{(-i)}(\lambda) = \big(J_\gamma^{(i)}(\lambda)\big)^{\angle} \tag{4.2}$$

For $i > 0$ we say that the space $J_\gamma^{(i)}(\lambda)$ is the *i-th osculating space of the curve J_γ at* λ.

Note that $J_\gamma = J_\gamma^{(0)}$. Directly from the definitions the subspaces $J_\gamma^{(i)}(\lambda)$ are coisotropic for $i > 0$ and isotropic for $i < 0$ and the tuple $\{J_\gamma^{(i)}(\lambda)\}_{i\in\mathbb{Z}}$ defines a filtration of W_γ. In other words, the curve $\lambda \mapsto \{J_\gamma^{(i)}(\lambda)\}_{i\in\mathbb{Z}}$ is a curve of flags of W_γ. Besides, it can be shown [23] that

$$\dim J^{(1)}(\lambda) - \dim J^{(0)}(\lambda) = \dim J^{(0)}(\lambda) - \dim J^{(-1)}(\lambda) = 1,$$

which in turn implies that $\dim J^{(i)}(\lambda) - \dim J^{(i-1)}(\lambda) \leq 1$, i.e. the jump of dimensions between the consecutive subspaces of the filtration $\{J_\gamma^{(i)}(\lambda)\}_{i\in\mathbb{Z}}$ is at most 1. This together with (3.6) implies that $\dim J_\gamma^{(i)}(\lambda) \leq n-3+i$ for $i>0$.

We say that λ is a *regular point of* $(D^2)^\perp \backslash (D^3)^\perp$ if $\dim J_\gamma^{(i)}(\lambda) = n-3+i$ for $0 < i \leq n-3$ or, equivalently, if $J_\gamma^{(n-3)}(\lambda) = W_\gamma$. A rank 2 distribution D is called of *maximal class at a point* $q \in M$ if at least one point in $\pi^{-1}(q) \cap (D^2)^\perp$ is regular. Since by (2.4) the characteristic distribution $\mathscr{C}$ generated by a vector field depending algebraically on the fibers $(D^2)^\perp$, if D is of maximal class at a point $q \in M$, then the set of all regular points of $\pi^{-1}(q) \cap (D^2)^\perp$ is non-empty open set in Zariski topology. The same argument is used to show that the set of germs of rank 2 distributions of maximal class is generic.

If D is of maximal class at q and $n \geq 5$, then by necessity $\dim D^3(q) = 5$. The following question is still open: Does there exist a rank 2 distribution with $\dim D^3 = 5$ such that it is not of maximal class on some open set of M? We proved that the answer is negative for $n \leq 8$ and we have strong evidences that the answer is negative in general.

Remark 2. Note that from (2.4) it follow that if a rank 2 distribution D is of maximal class at a point $q \in M$ then the set of all lines $\{d\pi\big(\mathscr{C}(\lambda)\big) : \lambda \in \mathscr{R}_D \cap \pi^{-1}(q)\}$ is an open and dense subset of the projectivization $\mathbb{P}D(q)$ of the plane $D(q)$, where, as before, $\pi : T^*M \to M$ is the canonical projection. □

From now on we will work with rank 2 distributions of maximal class. In this case $\dim J_\gamma^{(4-n)}(\lambda) = 1$, i.e. the curve $J_\gamma^{(4-n)}$ is a curve in the projective space $\mathbb{P}W_\gamma$. Moreover, the curve of flags $\lambda \mapsto \{J_\gamma^{(i)}(\lambda)\}_{i=3-n}^{n-3}$, $\lambda \in \gamma$ is the curve of complete flags and the space $J_\gamma^{(i)}(\lambda)$ is the $(i+n-4)$th-osculating space of the curve $J_\gamma^{(4-n)}$. In other words, the whole curve of complete flags $\lambda \mapsto \{J_\gamma^{(i)}(\lambda)\}_{i=3-n}^{n-3}$, $\lambda \in \gamma$ can be recovered from the curve $J_\gamma^{(4-n)}$ and the differential geometry of Jacobi curves of abnormal extremals of rank 2 distributions is reduced to the differential geometry of curves in projective spaces.

5 Canonical projective structure on curves in projective spaces

The differential geometry of curves in projective spaces is the classical subject, essentially completed already in 1905 by E.J. Wilczynski ([21]). In particular, it is well known that these curves are endowed with the canonical projective structure, i.e., there is a distinguished set of parameterizations (called projective) such that the transition function from one such parametrization to another is a Möbius transformation. Let us demonstrate how to construct it for the curve $\lambda \mapsto J_\gamma^{(4-n)}(\lambda)$, $\lambda \in \gamma$.

As before, let $C(J_\gamma^{(4-n)})$ be the tautological bundle $C(J_\gamma^{(4-n)})$ over $J_\gamma^{(4-n)}$. Set $m = n-3$. Here we use a "naive approach", based on reparametrization rules for certain coefficient in the expansion of the derivative of order $2m$ of certain sections of $C(J_\gamma^{(4-n)})$ w.r.t. to the lower order derivatives of this sections. For the more algebraic point of view, based on a Tanaka-like theory of curves of flags and $\mathfrak{sl}_2$-representations see [8, 12].

Take some parametrization $\psi\colon I \mapsto \gamma$ of γ, where I is an interval in $\mathbb{R}$. By above, for any section ℓ of $C(J_\gamma^{(4-n)})$ one has that

$$\operatorname{span}\Big\{\frac{d^j}{dt^j}\ell\big(\psi(t)\big) \mid 0 \le j \le 2m-1\Big\} = W_\gamma. \tag{5.1}$$

A curves in the projective space $\mathbb{P}W_\gamma$ satisfying the last property is called *regular* (or *convex*). It is well known that there exists the unique, up to the multiplication by a nonzero constant, section E of $C(J_\gamma^{(4-n)})$, called a *canonical section of* $C(J_\gamma^{(4-n)})$ *with respect to the parametrization* ψ, such that

$$\frac{d^{2m}}{dt^{2m}}E\big(\psi(t)\big) = \sum_{i=0}^{2m-2} B_i(t)\frac{d^i}{dt^i}E\big(\psi(t)\big), \tag{5.2}$$

i.e. the coefficient of the term $\frac{d^{2m-1}}{dt^{2m-1}}E\big(\psi(t)\big)$ in the linear decomposition of $\frac{d^{2m}}{dt^{2m}}E\big(\psi(t)\big)$ w.r.t. the basis $\big\{\frac{d^i}{dt^i}E\big(\psi(t)\big) : 0 \le i \le 2m-1\big\}$ vanishes.

Further, let ψ_1 be another parameter, $\widetilde{E}$ be a canonical section of $C(J_\gamma^{(4-n)})$ with respect to the parametrization ψ_1, and $\upsilon = \psi^{-1} \circ \psi_1$. Then directly from the definition it easy to see that

$$\widetilde{E}\big(\psi_1(\tau)\big) = c(\upsilon'(\tau))^{\frac{1}{2}-m}E(\psi(t)) \tag{5.3}$$

for some non-zero constant c.

Now let $\widetilde{B}_i(\tau)$ be the coefficient in the linear decomposition of $\frac{d^{2m}}{d\tau^{2m}}\widetilde{E}\big(\psi_1(\tau)\big)$ w.r.t. the basis $\big\{\frac{d^i}{d\tau^i}\widetilde{E}\big(\psi_1(t)\big) : 0 \le i \le 2m-1\big\}$ as in (5.2). Then, using the relation (5.3) it is not hard to show that the coefficients B_{2m-2} and $\widetilde{B}_{2m-2}$ in the decomposition (5.2), corresponding to parameterizations ψ and ψ_1, are related as follows:

$$\widetilde{B}_{2m-2}(\tau) = \upsilon'(\tau)^2 B_{2m-2}(\upsilon(\tau)) - \frac{m(4m^2-1)}{3}\mathbb{S}(\upsilon)(\tau), \tag{5.4}$$

where $\mathbb{S}(\upsilon)$ is the Schwarzian derivative of υ, $\mathbb{S}(\upsilon) = \frac{d}{d\tau}\Big(\frac{\upsilon''}{2\upsilon'}\Big) - \Big(\frac{\upsilon''}{2\upsilon'}\Big)^2$.

From the last formula and the fact that $\mathbb{S}\upsilon \equiv 0$ if and only if the function υ is Möbius it follows that *the set of all parameterizations* φ *of* γ *such that*

$$B_{2m-2} \equiv 0 \tag{5.5}$$

defines the canonical projective structure on γ. Such parameterizations are called the *projective parameterizations of the abnormal extremal* γ. If ψ and ψ_1 are two projective parametrizations, then there exists a Möbius transformation υ such that $\psi_1 = \psi \circ \upsilon$.

Note that the curve $J_\gamma^{(4-n)}$ is not an arbitrary regular curve in the projective space $\mathbb{P}W$. It satisfies the following additional property:

(S1) *The $(n-4)$th-osculating space of $J_\gamma^{(4-n)}$ at any point λ is Lagrangian.*

As shown already by Wilczynski [21] such curves are *self-dual* in the following sense:

(S2) *The curve $(J_\gamma^{(n-4)})^*$ in the projectivization $\mathbb{P}W_\gamma^*$ of the dual space W_γ^*, which is dual to the curve of hyperplanes $J_\gamma^{(n-4)}$ obtained from the original curve $J_\gamma^{(4-n)}$ by the osculation of order $2(n-4)$, is equivalent to the original curve $J_\gamma^{(4-n)}$, i.e. there is a linear transformation $A : W \mapsto W^*$ sending $J_\gamma^{(n-4)}$ onto $(J_\gamma^{(n-4)})^*$.*

Note that in contrast to property (S1) the formulation of property (S2) does not involve a symplectic structure on W_γ. However, it can be shown [21, 17] that if the property (S2) holds then there exists a unique, up to a multiplication by a nonzero constant, symplectic structure on W_γ such that the property (S1) holds (here it is important that $\dim W_\gamma$ is even; similar statement for the case of odd dimensional linear space involves nondegenerate symmetric forms instead of skew-symmetric ones). Since in our case the symplectic structure on W_γ is a priori given, in the sequel we will consider projective spaces of linear symplectic spaces only and by self-dual curves we will mean curves satisfying property (S1).

Using the coefficients of the decomposition (5.2) w.r.t. a projective parameter t one can construct the (relative) invariants of the unparametrized curve $J_\gamma^{(4-n)}$, called the Wilczynski invariants. Since we shall not use these invariants in the sequel, we will not give here their construction referring the interested reader to [8, 12]. Note only that in the case of a self-dual curve in such decomposition also $B_{2m-3}(t) \equiv 0$ and the first nontrivial Wilczynski invariant is $B_{2m-4}(t)dt^4$, i.e. this is the homogeneous function of degree 4 on each tangent line to our curve. As shown in [24], for rank 2 distributions in $\mathbb{R}^5$ with maximal possible small growth vector $(2,3,5)$, this invariant, calculated along each abnormal extremal, gives the classical Cartan invariant of [7].

6 Canonical frames for rank 2 distributions of maximal class

Now let $\mathscr{R}_D$ be the set if all regular points of $(D^2)^\perp \backslash (D^3)^\perp$. Denote by $\mathfrak{P}_\lambda$ the set of all projective parameterizations ψ on the characteristic curve γ, passing through λ, such that $\psi(0) = \lambda$. Let

$$\Sigma_D = \{(\lambda, \psi) : \lambda \in \mathscr{R}_D, \psi \in \mathfrak{P}_\lambda\}.$$

Actually, Σ_D is a principal bundle over $\mathscr{R}_D$ with the structural group of all Möbius transformations, preserving 0 and $\dim \Sigma_D = 2n-1$. The main results of [9, 10] can be summarized in the following:

Theorem 2. *For any rank 2 distribution in $\mathbb{R}^n$ with $n > 5$ of maximal class there exists the canonical, up to the action of $\mathbb{Z}_2$, frame on the corresponding $(2n-1)$-dimensional manifold Σ_D so that two distributions from the considered class are equivalent if and only if their canonical frames are equivalent. The group of symmetries of such distributions is at most $(2n-1)$-dimensional and this upper bound is sharp. All distributions from the considered class with $(2n-1)$-dimensional Lie algebra of infinitesimal symmetries is locally equivalent to the distribution $D_{((0,\dots 0)}$ generated by the vector fields X_1 and X_2 from* (1.3)-(1.4) *with all r_i equal to* 0 *or, equivalently, associated with the underdetermined ODE $z'(x) = \big(y^{(n-3)}(x)\big)^2$. The symmetry algebra of this distribution is isomorphic to a semidirect sum of $\mathfrak{gl}(2,\mathbb{R})$ and $(2n-5)$-dimensional Heisenberg algebra $\mathfrak{n}_{2n-5}$ such that $\mathfrak{gl}(2,\mathbb{R})$ acts irreducibly on a complement of the center of $\mathfrak{n}_{2n-5}$ to $\mathfrak{n}_{2n-5}$ itself.*

7 Canonical frames for rank 2 distributions of maximal class with distinguish parametrization on abnormal extremals

Let us show that for regular control systems on rank 2 distributions in the sense Definition 2 a special parametrization, up to a shift, can be distinguished on each abnormal extremal lying in $\mathscr{R}_D$. Let $\mathscr{V}_q$ be the set of the admissible velocities of the control system under consideration at the point $q \in M$. Let $\widehat{\mathscr{R}}$ be a subset of $\mathscr{R}_D$ consisting of all points λ such that the image under $d\pi$ of the tangent line at λ to the abnormal extremal passing through λ is a regular line in $D\big(\pi(\lambda)\big)$ in the sense of Definition 1 (here , as before $\pi : T^*M \to M$ is the canonical projection). Then by Definition 2 and Remark 2 the set $\widehat{\mathscr{R}}$ is a non-empty open subset of $(D^2)^\perp$. Given a regular line L in $D(q)$ let $w(L)$ be the admissible velocity in L of the smallest norm. Clearly $w(L)$ does not depend on the choice of a norm in $D(q)$, but in general it may be defined up to a sign (for example, in the sub-(pseudo) Riemannian case).

A parametrization $\psi : I \mapsto \gamma$ of an abnormal extremal γ living in $\widehat{\mathscr{R}}$ is called *weakly canonical (with respect to the regular control system given by the set of admissible velocities $\{\mathscr{V}_q\}_{q\in M}$)* if

$$d\pi\big(\frac{d}{dt}\gamma(\psi(t))\big) = w\Big(\operatorname{span} d\pi\big(\frac{d}{dt}\gamma(\psi(t))\big)\Big) \tag{7.1}$$

This parametrization is defined up to a shift and maybe up to the change of orientation. In the case when the orientation is not fixed by (7.1) we can fix the orientation as follows: Since the curve $J_\gamma^{(4-n)}$ is self-dual, given a parametrization ψ on γ, among all canonical sections of the tautological bundle $C(J_\gamma^{(4-n)})$ (defined up to the multiplication by a nonzero constant) there exists the unique, up to a sign, section E

of such that (5.2) holds and

$$\left|\tilde{\sigma}_\gamma\left(\frac{d^{n-3}}{dt^{n-3}}E\big(\psi(t)\big),\frac{d^{n-4}}{dt^{n-4}}E\big(\psi(t)\big)\right)\right|\equiv 1. \tag{7.2}$$

This section E will be called the *strongly canonical section* of $C(J_\gamma^{(4-n)})$ with respect to the parametrization ψ. The parametrization ψ is called the *canonical parametrization of the abnormal extremal* γ if (7.1) holds and

$$\tilde{\sigma}_\gamma\left(\frac{d^{n-3}}{dt^{n-3}}E\big(\psi(t)\big),\frac{d^{n-4}}{dt^{n-4}}E\big(\psi(t)\big)\right)\equiv 1. \tag{7.3}$$

We finally obtain the parametrization of γ defined up to a shift only.

Finally let $\widetilde{\mathscr{R}}$ be a subset of $\widehat{\mathscr{R}}$ where the vector field consisting of the tangent vectors to the abnormal extremals parameterized by the canonical parameter is smooth. Note that $\widetilde{\mathscr{R}}$ is an open and dense subset of $\widehat{\mathscr{R}}$. For affine control systems with one input and a non-zero drift and for sub-Riemannian structures $\widetilde{\mathscr{R}}$ coincides with the set $\mathscr{R}_D$ of the regular points in $(D^2)^\perp\backslash(D^3)^\perp$.

Note that the canonical parametrization is preserved by the homotheties of the fibers of $(D^2)^\perp$. Namely, if δ_s is the flow of homotheties on the fibers of T^*M: $\delta_s(p,q)=(e^sp,q),\quad q\in M,\ p\in T_q^*M$ or, equivalently, the flow generated by the Euler field e generates this flow, then $\psi: I\mapsto\gamma$ is the canonical parametrization on an abnormal extremal γ if and only if $\delta_s\circ\psi$ is the canonical parametrization on the abnormal extremal $\delta_s\circ\gamma$.

The main goal of this section is to prove the following

Theorem 3. *Given a regular control system on a rank 2 distribution D of maximal class one can assign to it a canonical, up to the action of $\mathbb{Z}_2$, frame on the set $\widetilde{\mathscr{R}}$ defined above so that two objects from the considered class are micro-locally equivalent if and only if their canonical frames are equivalent.*

Proof. First, let h be the vector field consisting of the tangent vectors to the abnormal extremals parameterized by the canonical parameter.

Second, given $\lambda\in(D^2)^\perp$ denote by $V(\lambda)$ the tangent space to the fiber of the bundle $\pi\colon (D^2)^\perp\mapsto M$ (the vertical subspace of $T_\lambda(D^2)^\perp$),

$$V(\lambda)=\{v\in T_\lambda(D^2)^\perp,\pi_*v=0\}. \tag{7.4}$$

It is easy to show ([10, 23]) that

$$d\phi\big(V(\lambda)\oplus\mathscr{C}(\lambda)\big)=J_\gamma^{(-1)}(\lambda)\quad \mathrm{mod}\,\mathbb{R}\bar{e}, \tag{7.5}$$

where ϕ is as in (3.2), $\bar{e}=\phi_*e$ with e being the Euler field, and γ is the abnormal extremal passing through λ. Define also the following subspaces of $T_\lambda(D^2)^\perp$:

$$\mathscr{J}^{(i)}(\lambda)=\{w\in T_\lambda(D^2)^\perp: d\phi(w)\in J_\gamma^{(i)}(\lambda)\,\mathrm{mod}\,\mathbb{R}\bar{e}\}. \tag{7.6}$$

Directly from the definition, if $\lambda \in \mathscr{R}_D$, then

$$[\mathscr{C}, \mathscr{J}^{(i)}](\lambda) = \mathscr{J}^{(i+1)}(\lambda). \tag{7.7}$$

Also, if $V^{(i)}(\lambda) = V(\lambda) \cap \mathscr{J}^{(i)}(\lambda)$, then

$$\mathscr{J}^{(i)}(\lambda) = V^{(i)}(\lambda) \oplus \mathscr{C}(\lambda) \quad \forall i \leq 0. \tag{7.8}$$

Moreover, it can be shown ([10, Lemma 2]) that

$$[V^{(i)}, V^{(i)}] \subseteq V^{(i)}, \quad [V^{(i)}, \mathscr{J}^{(i)}] \subseteq \mathscr{J}^{(i)}, \quad \forall i \leq 0. \tag{7.9}$$

Let E be the strongly canonical section of $C(J_\gamma^{(4-n)})$ with respect to the canonical parametrization ψ of the abnormal extremal γ (as defined by (7.2)). Then (7.5) implies that a vector field ε_1 such that

(A1) $d\phi\big(\varepsilon_1(\lambda)\big) \equiv E \mod \bar{e}$,
(A2) ε_1 is the section of the vertical distribution V

is defined modulo the Euler field e. Note that conditions (A1) and (A2) also imply that ε_1 is the section of $V^{(4-n)}$.

Lemma 1. *Among all vector fields ε_1 satisfying conditions (A1) and (A2), there exists the unique, up to a multiplication by -1, vector field such that*

$$\big[\varepsilon_1, [h, \varepsilon_1]\big](\lambda) \in \operatorname{span}\{e(\lambda), h(\lambda), \varepsilon_1(\lambda)\}. \tag{7.10}$$

Proof. Let $\tilde{\varepsilon}_1$ be a vector field satisfying the conditions (A1) and (A2). Then $\tilde{\varepsilon}_1$ is the section of $V^{(4-n)}$. Using (7.8) and (7.9) for $n > 5$ and also the definition of $\mathscr{J}$ given by (3.1) in the case $n = 5$, we get

$$\big[\tilde{\varepsilon}_1, [h, \tilde{\varepsilon}_1]\big] \equiv k[h, \tilde{\varepsilon}_1] \mod \operatorname{span}\{\mathrm{e}, \mathrm{h}, \tilde{\varepsilon}_1\} \tag{7.11}$$

for some function k. Now let ε_1 be another vector field satisfying conditions (A1) and (A2). Then by above there exists a function μ such that

$$\varepsilon_1 = \pm \tilde{\varepsilon}_1 + \mu e. \tag{7.12}$$

From the fact that the canonical parametrization is preserved by the homotheties of the fibers of $(D^2)^\perp$ it follows that $[e, h] = 0$. Also from the normalization condition (7.2) it is easy to get that

$$[e, \varepsilon_1] = -\frac{1}{2}\varepsilon_1 \mod \operatorname{span}(e). \tag{7.13}$$

Then

$$\big[e, [h, \varepsilon_1]\big] = -\frac{1}{2}[h, \varepsilon_1] \mod (e, h), \tag{7.14}$$

From this and (7.12) it follows that

$$\big[\varepsilon_1,[h,\varepsilon_1]\big] \equiv \big(k \mp \frac{\mu}{2}\big)[h,\varepsilon_1] \ \mathrm{span}\{e,h,\varepsilon_1\}, \tag{7.15}$$

which implies the statement of the lemma: the required vector $\tilde{\varepsilon}_1$ is obtained by taking $\mu = \pm 2k$. $\Box$

Now we are ready to construct the canonical frame on the set $\widetilde{\mathscr{R}}$. One option is to take as a canonical frame the following one:

$$\big\{e,h,\varepsilon_1,\{(\mathrm{ad}\,h)^i\varepsilon_1\}_{i=1}^{2n-7},[\varepsilon_1,(\mathrm{ad}\,h)^{2n-7}\varepsilon_1]\big\}, \tag{7.16}$$

where ε_1 is as in Lemma 1. Let us explain why it is indeed a frame. First the vector fields $\big\{e,h,\varepsilon_1,\{(\mathrm{ad}\,h)^i\varepsilon_1\}_{i=1}^{2n-7}\big\}$ are linearly independent on $\widetilde{\mathscr{R}}$ due to the relation (7.7). Besides $[\varepsilon_1,(\mathrm{ad}\,h)^{2n-7}\varepsilon_1](\lambda) \notin \mathscr{J}^{(n-3)}(\lambda)$. Otherwise, $\varepsilon_1(\lambda)$ belongs to the kernel of the form $\sigma(\lambda)|_{(D^2)^\perp}$ and therefore it must be collinear to h. We get a contradiction. Therefore the tuple of vectors in (7.16) constitute a frame on $\widetilde{\mathscr{R}}$.

The construction of the frame (7.16) is intrinsic. However, in order to guaranty that two objects from the considered class are equivalent if and only if their canonical frames are equivalent, we have to modify this frame such that it will contain the basis of the vertical distribution V (defined by 7.4). For this, replace the vector fields of the form $(adh)^i\varepsilon_1$ for $1 \le i \le n-4$ by their projections to $V^{(i)}$ with respect to the splitting (7.8), i.e. their vertical components with respect to this splitting. This completes the construction of the required canonical frame (defined up to the action of the required finite groups). The proof of Theorem 3 is completed.

As a direct consequence of Theorem 3 we have

Corollary 1. *For a regular control system on a rank 2 distribution D of maximal class the dimension of pseudo-group of micro-local symmetries does not exceed* $2n-3$.

8 Symplectic curvatures for the structures under consideration

Before proving Theorem 1 about the most symmetric models for geometric structures under consideration, we want to reformulate this theorem in more geometric terms. For this we distinguish special invariants for this structures called the *symplectic curvatures*. They are functions on the open subset $\widetilde{\mathscr{R}}$ of $\mathscr{R}_D$, defined in the beginning of the previous section.

From the construction of the previous section all curves $J_\gamma^{(4-n)}$ are parameterized by the canonical (up to a shift) parametrization ψ given by (7.1) (and maybe also by (7.3)). The geometry of parameterized regular self-dual curves in projective spaces is simpler than of unparametrized ones: instead of forms (relative invariants) on the curve we obtain invariant, which are scalar-valued function on the curve ([25]). The main result of [25] (Theorem 2 there) can be reformulated as follows (see also [17]): if E is a (strongly) canonical section of $C(J_\gamma^{(4-n)})$ with respect to the (canonical) parametrization ψ, then there exist m functions $\rho_1(t),\dots,\rho_m(t)$ such that

$$E^{(2m)}\big(\psi(t)\big) = \sum_{i=1}^{m}(-1)^{i+1}\frac{d^{m-i}}{dt^{m-i}}\Big(\rho_i(t)\frac{d^{m-i}}{dt^{m-i}}E\big(\psi(t)\big)\Big). \tag{8.1}$$

Note that formula (8.1) resembles the classical normal form for the formally self-adjoint linear differential operators [19][§1].

By constructions, the functions $\rho_1(t),\dots,\rho_m(t)$ are invariants of the parameterized curve $t\mapsto J_\gamma^{(4-n)}\big(\psi(t)\big)$ with respect to the action of the linear symplectic group on W_γ. We call the function $\rho_i(t)$ the *ith symplectic curvature of the parametrized curve* $t\mapsto J_\gamma^{(4-n)}\big(\psi(t)\big)$. Besides, the functions $\rho_1(t),\dots,\rho_m(t)$ constitute the fundamental system of symplectic invariant of the parametrized curve $t\mapsto J_\gamma^{(4-n)}\big(\psi(t)\big)$, i.e. they determine this curve uniquely up to a symplectic transformation. Moreover, these invariants are independent: for any tuple of m functions $\rho_1(t),\dots,\rho_m(t)$ on the interval $I\subseteq R$ there exists a parameterized regular self-dual curve $t\mapsto\Lambda(t)$, $t\in I$, in the projective space of dimension $2m-1$ with the ith symplectic curvature equal to $\rho_i(t)$ for any $1\le i\le m$.

Also in the sequel we will need the following

Remark 3. Assume that E is the strongly canonical section of $C(J_\gamma^{(4-n)})$ with respect to the parametrization ψ. Using the fact that the spaces $\operatorname{span}\Big\{\dfrac{d^j}{dt^j}E\big(\psi(t)\big)\Big\}_{j=1}^{m}$ are Lagrangian and the condition (7.2), it is easy to show that

$$\tilde{\sigma}_\gamma\Big(\frac{d^j}{dt^j}E\big(\psi(t)\big),\frac{d^i}{dt^i}E\big(\psi(t)\big)\Big)$$

are either identically equal to 0, if $i+j<2m-1$ or to ±1, if $i+j=2m-1$, or they are polynomial expressions (with universal constant coefficients) with respect to the symplectic curvatures $\rho_1(t),\dots,\rho_m(t)$ and their derivatives, if $i+j>2m$. $\Box$

Taking the ith symplectic curvature for Jacobi curves (parameterized by the canonical parameter) of all abnormal extremals living in $\widetilde{\mathscr{R}}$, we obtain the invariants of the regular control systems, called the *ith symplectic curvature* and denoted also by ρ_i. The symplectic curvatures are scalar valued functions on the set $\widetilde{\mathscr{R}}$.

9 The maximally symmetric models

Now we will find all structures from the considered classes having the pseudo-group of micro-local symmetries of dimension equal to $2n-3$. As a consequence of Corollary 1 if an object from the considered class has the pseudo-group of micro-local symmetries of dimension equal to $2n-3$ then all structure functions of the canonical frame (7.16) must be constant. Note that formula (8.1) can be rewritten in terms of the canonical frame (7.16) as follows

$$[h,\varepsilon_{2m}] = \sum_{i=1}^{m}(-1)^{i+1}(\mathrm{ad}\,h)^{m-i}\Big(\rho_i\big(\mathrm{ad}\,h^{m-i}\varepsilon_1\Big) \quad \mathrm{mod}\ \mathrm{span}\{\mathrm{e},\mathrm{h}\}, \tag{9.1}$$

where ρ_i are the ith symplectic curvatures of a structures under consideration. This implies that the symplectic curvatures of all order must be constant for any structure from the considered classes having $2n-3$-dimensional pseudo-group of micro-local symmetries . This implies that the following theorem is equivalent to Theorems 1

Theorem 4. *Given any tuples of* $n-3$ *numbers* $(r_1,\dots,r_{n-3})$ *there exists the unique, up to micro-local equivalence, regular control system on a rank 2 distribution of maximal class in* $\mathbb{R}^n$ *with* $n\geq 5$ *having the group of micro-local symmetries of dimension* $2n-3$ *and the ith symplectic curvature identically equal to* r_i *for any* $1\leq i\leq n-3$. *Such regular control system is micro-locally equivalent to the system* $A_{(r_1,\dots,r_{n-3})}$ *defined by* (1.2)-(1.4).

Proof. First, let us prove the uniqueness. Take a structure from the considered class having the pseudo-group of micro-local symmetries of dimension $2n-3$ and the ith symplectic curvature identically equal to r_i for any $1\leq i\leq m$, where, as before, $m=n-3$. Then, as was already mentioned, all structure functions of the canonical frame (7.16) must be constant. The uniqueness will be proved if we will show that all nontrivial structure function (i.e. those that are not prescribed by the normalization conditions for the canonical frame) are uniquely determined by the tuple $(r_1,\dots,r_{n-3})$.

Let ε_1 be as in the Lemma 1. Denote

$$\varepsilon_{i+1} := (\mathrm{ad}\,h)^i\varepsilon_1, \quad \nu=[\varepsilon_1,\varepsilon_{2m}] \tag{9.2}$$

In this notations the canonical frame (7.16) is $\{e,h,\varepsilon_1,\dots,\varepsilon_{2m},\eta\}$.

1. Let us prove that

$$[e,\varepsilon_1] = -\frac{1}{2}\varepsilon_1 \tag{9.3}$$

where, as before e is the Euler field. Indeed, from (7.14)

$$[e,\varepsilon_1] = -\frac{1}{2}\varepsilon_1 + ae \tag{9.4}$$

where a is constant by our assumptions. Then, using the Jacobi identity and the fact that

$$[e,h]=0 \tag{9.5}$$

we get that

$$[e,\varepsilon_2] = \big[e,[h,\varepsilon_1]\big] = \big[h,[e,\varepsilon_1]\big] = [h,-\frac{1}{2}\varepsilon_1+ae] = -\frac{1}{2}\varepsilon_2 \tag{9.6}$$

Further, from the normalization condition (7.10) and formula (9.4) it follows that

$$\big[e,[\varepsilon_1,\varepsilon_2]\big] \in \operatorname{span}\{e(\lambda),h(\lambda),\varepsilon_1(\lambda)\} \tag{9.7}$$

On the other hand, using the Jacobi identity and formulas (9.4),(9.5),(9.6), we get that

$$\big[e,[\varepsilon_1,\varepsilon_2]\big] = \big[[e,\varepsilon_1],\varepsilon_2\big] + \big[\varepsilon_1,[e,\varepsilon_2]\big] = [-\frac{1}{2}\varepsilon_1 + ae,\varepsilon_2] -$$
$$\frac{1}{2}[\varepsilon_1,\varepsilon_2] \equiv -\frac{a}{2}\varepsilon_2 \mod \operatorname{span}\{e(\lambda),h(\lambda),\varepsilon_1(\lambda)\},$$

which together with (9.7) implies that $a=0$.

2. By analogy with the chain of the equalities (9.7) we can prove that

$$[e,\varepsilon_i] = -\frac{1}{2}\varepsilon_i, \quad \forall 1 \le i \le 2m, \tag{9.8}$$

which in turn implies by the Jacobi identity that

$$\big[e,[\varepsilon_i,\varepsilon_j]\big] = -[\varepsilon_i,\varepsilon_j], \quad \forall 1 \le i,j \le 2m. \tag{9.9}$$

In particular, $[e,\eta] = -\eta$.

3. Let us show that

$$[h,\varepsilon_{2m}] = \sum_{i=1}^{m-1}(-1)^{i+1} r_i \varepsilon_{2(m-i)} \tag{9.10}$$

From (9.1) and our assumptions it follows that

$$[h,\varepsilon_{2m}] = \sum_{i=1}^{m-1}(-1)^{i+1} r_i \varepsilon_{2(m-i)} + \gamma e + \delta h \tag{9.11}$$

for some constants γ and δ. Applying $\operatorname{ad} e$ to both sides of (9.11) and using the Jacobi identity and formulas (9.5) and (9.8), we will get that $\gamma = \delta = 0$, which implies (9.11).

4. Let us prove that

$$[\varepsilon_i,\varepsilon_j] = d_{ij}\eta \tag{9.12}$$

for some constants d_{ij} Indeed, in general

$$[\varepsilon_i,\varepsilon_j] = b_{ij}e + c_{ij}h + d_{ij}\eta + \sum_{k=1}^{2m} a_{ij}^k \varepsilon_k \tag{9.13}$$

where a_{ij}^k, b_{ij}, c_{ij} and d_{ij} are constant by our assumptions. Applying $\operatorname{ad} e$ to both sides of (9.13) and using the Jacobi identity and the formulas (9.5), (9.8), and (9.9), we get

$$-[\varepsilon_i,\varepsilon_j] = -d_{ij}\eta - \frac{1}{2}\sum_{k=1}^{2m} a_{ij}^k \varepsilon_k \tag{9.14}$$

Comparing (9.13) and (9.14) we get that $a_{ij}^k = b_{ij} = c_{ij} = 0$, which implies (9.12).

5. Moreover, by Remark 3 and the definition of the vector field η (see (9.2)) the constants d_{ij} from (9.12) are either identically equal to 0, if $i+j<2m$ or equal to $(-1)^{i-1}$, if $i+j=2m+1$, or they are polynomial expressions (with universal constant coefficients) with respect to the constant symplectic curvatures $r_1 \dots, r_m$, if $i+j>2m$.
6. The remaining brackets of the canonical frame are obtained iteratively from the brackets considered in the previous items.

Therefore all nontrivial structure functions of the canonical frame are determined by the tuple $(r_1,\dots,r_{n-3})$, which completes the proof of uniqueness.

To prove the existence one checks by the direct computations that the models $A_{(r_1,\dots,r_m)}$ have the prescribed symplectic curvatures and that all structure functions of their canonical frame are constant similarly to the proof of the existence part of Theorem 3 in [10], devoted to the computation of the canonical frame for $D_{(0,\dots,0)}$. $\square$

Remark 4. As a matter of fact it can be shown that Theorem 3 (with a modified set $\widetilde{\mathscr{R}}$), Corollary 1, and Theorem 4 are true if we replace the regularity condition for control systems given in Definition 2 by the following weaker one: for any point q the curve of admissible velocities $\mathscr{V}_q$ does not belong entirely to a line through the origin. One only needs more technicalities in the description of the set $\widetilde{\mathscr{R}}$ in Theorem 3. $\square$.

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