A CONGRUENCE MODULO FOUR IN REAL SCHUBERT CALCULUS

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Abstract. We establish a congruence modulo four in the real Schubert calculus on the Grassmannian of m-planes in $2m$-space. This congruence holds for fibers of the Wronski map and a generalization to what we call symmetric Schubert problems. This strengthens the usual congruence modulo two for numbers of real solutions to geometric problems. It also gives examples of geometric problems given by fibers of a map whose degree is zero but where each fiber contains real points.

Introduction

The number of real solutions to a system of real equations is congruent to the number of complex solutions modulo two, for the simple reason that complex conjugation is an involution which acts freely on the nonreal solutions. We establish an additional congruence modulo four for certain Schubert problems on the Grassmannian of m-planes in the space of polynomials of degree at most $2m-1$, for $m > 2$, which were originally observed in a computational experiment when $m = 3$. These Schubert problems come from the Wronski map. The reason for this congruence is that a natural symplectic structure on this space of polynomials induces an additional geometric involution on this Grassmannian which commutes with the Wronski map. This key result (Lemma 9) is generalized in a sequel to this paper [10].

This second involution commutes with complex conjugation, and the group they generate consists of the identity and three involutions whose fixed points are, respectively, the real Grassmannian, the Lagrangian Grassmannian, and a twisted real form of the Grassmannian, which we call the Hermitian Grassmannian. The common fixed point locus of this group is the real Lagrangian Grassmannian, and the congruence modulo four is a consequence of it not forming a hypersurface in the real Grassmannian when $m > 2$. This congruence is a general fact (Lemma 6) concerning fibers of a real map that has a second involution, when its fixed point locus has codimension at least two. It also applies to what we call symmetric Schubert problems which have the codimension condition on their Lagrangian locus. While we are unable to characterize which symmetric Schubert problems enjoy this condition, we are able to show this condition for a large class of symmetric Schubert problems.

Let $\mathbb{K}$ be a field which will either be the real numbers, $\mathbb{R}$, or the complex numbers, $\mathbb{C}$. We write $\mathbb{K}_d[t]$ for the $d+1$ dimensional vector space of univariate polynomials of degree

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at most d with coefficients from $\mathbb{K}$. Given $f_1,\dots,f_m \in \mathbb{K}_{m+p-1}[t]$, their Wronskian is the determinant

$$\mathrm{Wr}(f_1,\dots,f_m) \ :=\ \det\begin{pmatrix} f_1 & f_2 & \cdots & f_m \\ f_1' & f_2' & \cdots & f_m' \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(m-1)} & f_2^{(m-1)} & \cdots & f_m^{(m-1)} \end{pmatrix},$$

which is a polynomial of degree at most mp.

Replacing $f_1,\dots,f_m$ by polynomials $g_1,\dots,g_m$ with the same linear span will change their Wronskian by a constant, which is the determinant of the matrix expressing the g_i in terms of the f_j. Thus the Wronskian is a well-defined map

$$\mathrm{Wr}\ :\ \mathrm{Gr}(m,\mathbb{K}_{m+p-1}[t]) \ \longrightarrow\ \mathrm{Gr}(1,\mathbb{K}_{mp}[t]) \ =\ \mathbb{P}(\mathbb{K}_{mp}[t])\,,$$

where $\mathrm{Gr}(k,V)$ is the Grassmannian of k-dimensional subspaces of the $\mathbb{K}$-vector space V. Both $\mathrm{Gr}(m,\mathbb{K}_{m+p-1}[t])$ and $\mathbb{P}(\mathbb{K}_{mp}[t])$ are algebraic manifolds of dimension mp, and Wr is a finite map.

When $\mathbb{K}=\mathbb{C}$, the degree of the Wronski map, which is the number of points in a fiber above a regular value, was shown by Eisenbud and Harris [1] (based on earlier work of Schubert [21]) to be

$$\#^G_{m,p} \ :=\ \frac{(mp)!\cdot 1!2!\cdots(p{-}1)!}{m!(m{+}1)!\cdots(m{+}p{-}1)!}\,.$$

The real version of this *inverse Wronski problem*, namely identifying the real subspaces of polynomials in the fiber above a real polynomial $\Phi\in\mathbb{R}_{mp}[t]$, has been the subject of recent interest. This began with the conjecture of Boris Shapiro and Michael Shapiro (*c.* 1994), who conjectured that if Φ had all of its roots real, then every (*a priori*) complex vector space in the fiber $\mathrm{Wr}^{-1}(\Phi)$ would be real. Significant evidence for this conjecture, both theoretical and computational, was found in [24]. Eremenko and Gabrielov [4] proved the conjecture when $\min\{m,p\}=2$, and it was proved for all m,p by Mukhin, Tarasov, and Varchenko [15, 16]. In their study of this conjecture, Eremenko and Gabrielov [2] calculated the topological degree of the real Wronski map, obtaining a nonzero lower bound for the number of real subspaces in the fiber above a general real polynomial of degree mp, when $m{+}p$ is odd. When $m{+}p$ is even, the topological degree is zero. When both m and p are even, they gave a real polynomial of degree mp with no real preimages under the Wronski map [3]. It remains open when both m and p are odd whether or not there is a nontrivial lower bound on the number of real subspaces in the fibers of the Wronski map. When $m=p$, we establish a congruence modulo four on the number of real points in a fiber of the Wronski map.

Theorem 1. *Suppose that $m=p$ and $m\geq 3$. For every $\Phi(t)\in\mathbb{R}_{m^2}[t]$, the number of real points counted with multiplicity in $\mathrm{Wr}^{-1}(\Phi(t))$ is congruent to $\#^G_{m,m}$, modulo four.*

Since $\#^G_{3,3}=42$, which is congruent to 2 modulo four, we obtain the following corollary.

Corollary 2. *When $m=p=3$ there will always be at least two real subspaces counted with multiplicity in the fiber $\mathrm{Wr}^{-1}(\Phi(t))$ of a real polynomial $\Phi(t)$ of degree 9.*

Since the topological degree of the Wronski map $\mathrm{Wr}\colon \mathrm{Gr}(3, \mathbb{R}_5[t]) \to \mathbb{P}(\mathbb{R}_9[t])$ is zero, this corollary shows that the lower bound can be larger than the topological degree. The lower bound of 2 from Corollary 2 is sharp as we have examples of polynomials $\Phi(t)$ of degree nine having only two real three-dimensional subspaces of degree five polynomials in $\mathrm{Wr}^{-1}(\Phi(t))$. Table 1 shows the result of a computing $1,000,000$ fibers of this Wronski map, which consumed 391 gigaHertz-days of computing. The columns are labeled by the

Table 1. Fibers of the Wronski map.

Num. real	0	2	4	6	8	10	12	14	16	18	20	22
Frequency	0	66380	0	310667	0	208721	0	51774	0	34524	0	45940

Num. real	24	26	28	30	32	34	36	38	40	42	Total
Frequency	0	8560	0	17881	0	6632	0	771	0	248150	1000000

possible numbers of real solutions and each cell records how many computed instances had that number of real solutions.

Many of the symmetric Schubert problems treated in Section 4 also have a lower bound of two on their number of real solutions, coming from the congruence modulo four. In Example 17 we give a family of problems generalizing that of Corollary 2 and Table 1 which come from fibers of a map whose topological degree is zero, but which there are always at least two points in every fiber, showing that these topologically derived lower bounds may not be sharp.

There is now a growing body of examples of geometric problems which have lower bounds on their numbers of real solutions. This phenomenon occurs not only in the Schubert calculus [2], but also in counting rational curves on varieties [28, 12, 13] and lines on hypersurfaces of degree $2n-1$ in $\mathbb{P}^n$ [17, 5]. While there are some general theoretical bases for these lower bounds [18, 22], this phenomena is far from being understood. Two of us are conducting a large computational experiment of related lower bounds in the (not necessarily symmetric) Schubert calculus [7].

This paper is organized as follows. In Section 1 we present some basics on Schubert calculus and derive the canonical symplectic form on $\mathbb{K}_{2m-1}[t]$. We establish a framework for congruences modulo four in Section 2. In Section 3 we prove Theorem 1, and in Section 4 we extend this congruence to certain symmetric Schubert problems.

1. Definitions

All of our varieties and maps between varieties are defined over the real numbers. That is, they are complex varieties equipped with an antiholomorphic involution which we call complex conjugation. We will often write X when we intend its set of complex points, $X(\mathbb{C})$. Let $X(\mathbb{R})$ be the points of $X(\mathbb{C})$ which are fixed under complex conjugation. Write $\mathbb{Z}_2$ for the group $\mathbb{Z}/2\mathbb{Z}$ with two elements and $[n]$ for the set $\{1, 2, \ldots, n\}$ where n is a positive integer. We write V^* for the linear dual of a vector space V.

1.1. **Schubert Calculus.** Let V be a vector space over $\mathbb{K}$ of dimension $m+p$ where m,p are positive integers. We write $\mathrm{Gr}(m,V)$ for the Grassmannian of m-dimensional linear subspaces of V. This equivalently parametrizes m-dimensional quotients of V^*. This Grassmannian is a manifold of dimension mp and is a homogeneous space for the special linear group $SL(V)$.

A *(complete) flag* is a sequence $F_\bullet\colon F_1 \subset F_2 \subset \cdots \subset F_{m+p} = V$ of linear subspaces V with $\dim F_i = i$. A *partition* λ is a weakly decreasing sequence of integers $\lambda\colon p \geq \lambda_1 \geq \cdots \geq \lambda_m \geq 0$. A flag $F_\bullet$ and a partition λ together determine a *Schubert variety*

$$X_\lambda F_\bullet \ := \ \{H \in \mathrm{Gr}(m,V) \mid \dim H \cap F_{p+i-\lambda_i} \geq i\,, \quad i = 1,\dots,m\}\,. \tag{1.1}$$

This has codimension $|\lambda| := \lambda_1 + \cdots + \lambda_m$ in $\mathrm{Gr}(m,V)$. When $\lambda = (1,0,\dots,0)$ (written $\square$), the Schubert variety is

$$X_{\square} F_\bullet \ = \ \{H \in \mathrm{Gr}(m,V) \mid \dim H \cap F_p \geq 1\}\,.$$

That is, those H which meet F_p nontrivially.

A *Schubert problem* is a list $\boldsymbol{\lambda} = (\lambda^1,\dots,\lambda^n)$ of partitions where $|\lambda^1| + \cdots + |\lambda^n| = mp$. Given a Schubert problem $\boldsymbol{\lambda}$ and general flags $F^1_\bullet,\dots,F^n_\bullet$, the intersection of Schubert varieties

$$X_{\lambda^1}F^1_\bullet \cap X_{\lambda^2}F^2_\bullet \cap \cdots \cap X_{\lambda^n}F^n_\bullet$$

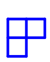
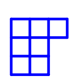
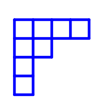
is transverse [14]. The number, $d(\boldsymbol{\lambda})$, of points in this intersection does not depend upon the choice of general flags and may be computed using algorithms from the Schubert calculus. Our concern here is not in computing this number, but in congruences satisfied by numbers of real solutions, for some Schubert problems and special choices of flags.

Suppose that $p = m$ and that V is equipped with a nondegenerate alternating (*symplectic*) form $\langle\cdot,\cdot\rangle$. The symplectic group is $\mathrm{Sp}(V)$ is the subgroup of $\mathrm{SL}(V)$ consisting of linear transformations which preserve this form $\langle\cdot,\cdot\rangle$,

$$\mathrm{Sp}(V) \ := \ \{g \in \mathrm{SL}(V) \mid \langle gv, gw\rangle = \langle v,w\rangle \ \forall v,w \in V\}\,.$$

A subspace $H \in \mathrm{Gr}(m,V)$ is *Lagrangian* if $\langle H,H\rangle \equiv 0$. The subset $\mathrm{LG}(V)$ of $\mathrm{Gr}(m,V)$ consisting of Lagrangian subspaces forms a manifold of dimension $\binom{m+1}{2}$ and is a homogeneous space for the symplectic group $\mathrm{Sp}(V)$.

The Lagrangian Grassmannian also has Schubert varieties. These require *isotropic flags*, which are flags where the subspace F_i is the annihilator of F_{2m-i} in that $\langle F_i, F_{2m-i}\rangle \equiv 0$. In particular, F_m is Lagrangian. We also need symmetric partitions, which we now explain. A partition $\lambda\colon m \geq \lambda_1 \geq \cdots \geq \lambda_m \geq 0$ may be represented by its Young diagram, which is an array of boxes with λ_i boxes in row i. For example,

$(2,1) \longleftrightarrow$ , $\qquad (3,2,2) \longleftrightarrow$ , $\qquad$ and $\qquad (4,2,1,1) \longleftrightarrow$ .

A partition is *symmetric* if its Young diagram is symmetric about its main diagonal. The partitions $(2,1)$ and $(4,2,1,1)$ are symmetric and $(3,2,2)$ is not symmetric.

A symmetric partition λ and an isotropic flag $F_\bullet$ together determine a Schubert variety of $\mathrm{LG}(V)$,

$$Y_\lambda F_\bullet \ := \ X_\lambda F_\bullet \cap \mathrm{LG}(V) \ = \ \{H \in \mathrm{LG}(V) \mid \dim H \cap F_{m+i-\lambda_i} \geq i\,, \quad i = 1,\dots,m\}\,.$$

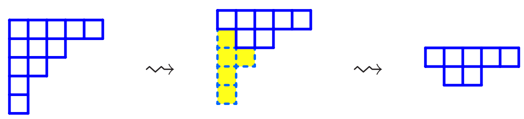
Its codimension in $\mathrm{LG}(V)$ is

$$\|\lambda\| \ := \ \tfrac{1}{2}(|\lambda| + \ell(\lambda))\,,$$

where $\ell(\lambda)$ is the number of boxes in the Young diagram of λ which lie on its main diagonal, $\ell(\lambda) = \max\{i \mid i \leq \lambda_i\}$. For example,

$\ell(\square)$ = $\ell($$)$ = 1 and $\ell($$)$ = $\ell($$)$ = 2.

We compare this to the alternative indexing set by strict partitions κ, which are strictly decreasing sequences of positive integers $\kappa\colon m \geq \kappa_1 > \cdots > \kappa_k > 0$. Such a sequence is obtained from a symmetric partition λ as the subsequence of positive numbers in the decreasing sequence $\lambda_1 > \lambda_2 - 1 > \cdots > \lambda_k - k + 1$. The diagram of a strict partition is obtained by removing the boxes below the diagonal from the diagram of a symmetric partition. For example,

$(5,3,2,1,1) \leftrightarrow$  $\leftrightarrow (5>2)\,.$

One value of this alterative indexing is that $\|\lambda\|$ is the number of boxes in the corresponding strict partition.

Given general isotropic flags $F^1_\bullet, \ldots, F^n_\bullet$ and symmetric partitions $\lambda^1, \ldots, \lambda^n$, the intersection of Schubert varieties

$$Y_{\lambda^1} F^1_\bullet \cap Y_{\lambda^2} F^2_\bullet \cap \cdots \cap Y_{\lambda^n} F^n_\bullet \tag{1.2}$$

is generically transverse [14]. When $n = 2$, we only need $F^1_\bullet$ and $F^2_\bullet$ to be in *linear general position*, $F^1_i \cap F^2_{2m-i} = \{0\}$, for $i = 1, \ldots, 2m-1$. In particular, the intersection (1.2) is either empty or every component has dimension

$$\binom{m+1}{2} - \|\lambda^1\| - \|\lambda^2\| - \cdots - \|\lambda^n\|\,.$$

This intersection consists of the Lagrangian subspaces in the corresponding intersection of Schubert varieties in the Grassmannian $\mathrm{Gr}(m, V)$.

1.2. **Canonical symplectic form on $\mathbb{K}_{2m-1}[t]$.** We follow the discussion of apolarity in §1 of [19]. Let U be a vector space over $\mathbb{K}$ and r a nonnegative integer. Write S^rU for the r-th symmetric power of U. Its elements are degree r homogeneous polynomials on U^*, the vector space dual to U, and thus r-forms on $\mathbb{P}(U)$, the projective space of hyperplanes of U. The dual vector space to S^rU is S^rU^*, whose elements act as differential operators of degree r on the polynomials in S^rU.

When $\dim(U) = 2$, the exterior product gives a symplectic form on U, $\langle u, v\rangle = u \wedge v$, which is well-defined up to a scalar (corresponding to an identification of $\wedge^2 U$ with $\mathbb{K}$). This induces a nondegenerate form $\langle\cdot,\cdot\rangle$ on S^rU which is well-defined up to a scalar multiple. It is symmetric when r is even and alternating when r is odd. Indeed, let s, t span U with $\langle s, t\rangle = 1$ and suppose that $u = (u_0 s + u_1 t)^r$ and $v = (v_0 s + v_1 t)^r$. Then a direct calculation gives $\langle u, v\rangle = (u_0 v_1 - v_0 u_1)^r$, so that $\langle u, v\rangle = (-1)^r \langle v, u\rangle$. Then the claim about the symmetry of the form follows as S^rU is spanned by r-th powers of linear forms. This computation gives the following formula: when $u = \sum_{i=0}^{r} u_i s^{r-i}t^i/i!$ and $v = \sum_{i=0}^{r} v_i s^{r-i}t^i/i!$, then, up to a scalar we have

$$\langle u, v\rangle \ = \ \sum_{i=0}^{r}(-1)^i u_i v_{r-i}\,. \tag{1.3}$$

Henceforth we dehomogenize, setting $s = 1$ and identifying S^rU with $\mathbb{K}_r[t]$ and work with these coordinates for $\mathbb{K}_r[t]$. We will restrict to the case when r is odd, writing $r = 2m-1$, and thus $\mathbb{K}_{2m-1}[t]$ has a natural structure of a symplectic vector space.

This symplectic form on $\mathbb{K}_{2m-1}[t]$ is classical and can be derived in several different ways. One attractive derivation is a consequence of $\mathbb{K}_{2m-1}[t]$ being the space of solutions of the self-adjoint linear differential equation $y^{(2m)} = 0$. In fact, many of the results of this paper may be generalized to Schubert calculus on spaces of solutions of a self-adjoint linear differential operator [10].

1.3. **Lagrangian involution and the Hermitian Grassmannian.** Suppose that $V \simeq \mathbb{C}^{2m}$ is a symplectic vector space with symplectic form $\langle\cdot,\cdot\rangle$. This form induces an involution on $\mathrm{Gr}(m, V)$ whose set of fixed points is the Lagrangian Grassmannian. For a linear subspace K of V, let

$$K^\angle \ := \ \{v \in V \mid \langle u, v\rangle = 0 \text{ for all } u \in K\}\,.$$

We have $(K^\angle)^\angle = K$ and $\dim K + \dim K^\angle = 2m$.

By this dimension calculation, $\angle$ restricts to an involution on $\mathrm{Gr}(m, V)$, which we call the *Lagrangian involution*. Since H is Lagrangian if and only if $H^\angle = H$, the Lagrangian Grassmannian is the set of fixed points of the Lagrangian involution,

$$\mathrm{LG}(V) \ = \ \mathrm{Gr}(m, V)_\angle\,.$$

The real points (those fixed by complex conjugation, $x \mapsto \overline{x}$) of $\mathrm{Gr}(m, V)$ and $\mathrm{LG}(V)$ are, respectively, the real Grassmannian $\mathbb{R}\mathrm{Gr}(m, V)$ and the real Lagrangian Grassmannian, $\mathbb{R}\mathrm{LG}(V)$. These real points correspond to real linear subspaces of V and real Lagrangian subspaces of V, respectively.

There is another distinguished type of linear subspace of V. An m-dimensional linear subspace H of V is *Hermitian* if

$$\overline{H} \ = \ H^\angle \qquad \text{equivalently} \qquad H \ = \ \overline{H}^\angle\,.$$

The *Hermitian Grassmannian* $\mathrm{HG}(V) \subset \mathrm{Gr}(m, V)$ is the set of all Hermitian linear subspaces of V. The map $H \mapsto \overline{H}^\angle$ is an anti-holomorphic involution which equips $\mathrm{Gr}(m, V)$ with a second real structure whose real points constitute the Hermitian Grassmannian. Thus $\mathrm{HG}(V)$ is a real algebraic manifold of dimension m^2 whose complexification is

$\mathrm{Gr}(m, V)$. We have the following diagram of inclusions.

$$\begin{tikzcd}[column sep=small]
 & \mathrm{Gr}(m,V) \arrow[dl, no head] \arrow[d, no head] \arrow[dr, no head] & \\
\mathbb{R}\mathrm{Gr}(m,v) \arrow[dr, no head] & \mathrm{LG}(V) \arrow[d, no head] & \mathrm{HG}(V) \arrow[dl, no head] \\
 & \mathbb{R}\mathrm{LG}(V) &
\end{tikzcd}$$

Proposition 3. *Each of these five Grassmannians may be realized as smooth compactifications of spaces of matrices according to the following table.*

Space of matrices	$M_{m\times m}(\mathbb{C})$	$M_{m\times m}(\mathbb{R})$	$Sym_m(\mathbb{C})$	$Sym_m(\mathbb{R})$	$\mathcal{H}_m$
Grassmannian	$\mathrm{Gr}(m,V)$	$\mathbb{R}\mathrm{Gr}(m,V)$	$\mathrm{LG}(V)$	$\mathbb{R}\mathrm{LG}(V)$	$\mathrm{HG}(V)$

Here $M_{m\times m}(\mathbb{K})$ is the set of all $m \times m$ matrices over $\mathbb{K}$, $Sym_m(\mathbb{K})$ is $m \times m$ symmetric matrices over $\mathbb{K}$, and $\mathcal{H}_m$ is all $m \times m$ Hermitian matrices.

Proof. Let $V \simeq \mathbb{C}^{2m}$ have ordered basis $\mathbf{e}_0, \ldots, \mathbf{e}_{2m-1}$ with the symplectic form

$$\langle \mathbf{e}_i \,,\, \mathbf{e}_j \rangle \;=\; (-1)^i \delta_{i,j}\,,$$

where $\delta_{i,j}$ is the Kronecker delta. This form will reappear in Section 3.

We use instead the ordered basis $\mathbf{f}_0, \ldots, \mathbf{f}_{2m-1}$ for V where

$$\mathbf{f}_i \;=\; \begin{cases} (-1)^i \mathbf{e}_i & \text{if } i < m \\ \mathbf{e}_{3m-1-i} & \text{if } i \geq m \end{cases} .$$

For $m = 5$, we have $(\mathbf{f}_0, \ldots, \mathbf{f}_{2m-1}) = (\mathbf{e}_0, -\mathbf{e}_1, \mathbf{e}_2, -\mathbf{e}_3, \mathbf{e}_4\,, \mathbf{e}_9, \mathbf{e}_8, \mathbf{e}_7, \mathbf{e}_6, \mathbf{e}_5)$. A simple calculation shows that

$$\langle \mathbf{f}_i \,,\, \mathbf{f}_j \rangle \;=\; \begin{cases} \delta_{i+m,j} & \text{if } i < m \\ -\delta_{i,j+m} & \text{if } i \geq m \end{cases} .$$

Let I_m be the $m \times m$ identity matrix and consider the map

$$\begin{array}{rccl} \rho : & M_{m\times m}(\mathbb{C}) & \hookrightarrow & \mathrm{Gr}(m,V) \\ & X & \longmapsto & \text{row space } [I_m \,:\, X]\,, \end{array}$$

where the columns of the matrix $[I_m : X]$ correspond to the ordered basis $(\mathbf{f}_0, \ldots, \mathbf{f}_{2m-1})$. It is classical that $\rho(M_{m\times m}(\mathbb{C}))$ is dense in $\mathrm{Gr}(m, V)$—it is the big Schubert cell. This establishes the first column of the table in the statement of the proposition, and the second column follows by restriction of scalars from $\mathbb{C}$ to $\mathbb{R}$.

Let us consider the map ρ in coordinates. Index the rows and columns of matrices in $M_{m\times m}$ by the numbers $0, 1, \ldots, m-1$. Let $X = (x_{i,j})_{i,j=0,\ldots,m-1} \in M_{m\times m}$. Then

$$\rho(X) \;=\; \mathrm{span}\Big\{\mathbf{f}_i \;+\; \sum_{a=0}^{m-1} x_{i,a}\mathbf{f}_{a+m} \;\mid\; i = 0, \ldots, m-1\Big\}\,.$$

Let $v_i(X)$ be the ith vector in this basis for $\rho(X)$, $v_i(X) = \mathbf{f}_i + \sum_a x_{i,a}\mathbf{f}_{a+m}$. Suppose that $\rho(X)^\angle = \rho(Y)$ for $X, Y \in M_{m\times m}(\mathbb{C})$. Then for each i and j,

$$0 \;=\; \langle v_i(X)\,,\, v_j(Y)\rangle \;=\; \Bigl\langle \mathbf{f}_i + \sum_a x_{i,a}\mathbf{f}_{a+m}\ ,\ \mathbf{f}_j + \sum_b y_{i,b}\mathbf{f}_{b+m}\Bigr\rangle \;=\; y_{j,i} \;-\; x_{i,j}\,.$$

Thus $\rho(X)^\angle = \rho(X^T)$, where X^T is the transpose of $X \in M_{m\times m}$. So $\rho(X)$ is Lagrangian if and only if X is symmetric, which implies the third and fourth columns in the table. Finally, we see that $\overline{\rho(X)}^\angle = \rho(\overline{X}^T)$, so that $\rho(X)$ is Hermitian if and only if X is Hermitian. □

Remark 4. Proposition 3 implies the assertions about the dimensions of these Grassmannians, which are equal to the dimensions of the corresponding spaces of matrices.

Remark 5. If we consider the symmetric form on $V = \mathbb{C}^{2m}$ given by $\langle \mathbf{f}_i, \mathbf{f}_j\rangle = \delta_{|i-j|,m}$, then the space of isotropic m-planes in V—called the orthogonal Grassmannian—corresponds to skew-symmetric matrices, those with $X^T = -X$, and there is a similar skew-Hermitian Grassmannian corresponding to skew-Hermitian matrices, those with $\overline{X}^T = -X$.

2. A very simple lemma

Suppose that $f\colon X \to Z$ is a proper dominant map between irreducible real varieties of the same dimension with Z smooth. Then f has a degree, d, which is the number of (complex) inverse images of a regular value $z \in Z$. When $z \in Z(\mathbb{R})$, we have the congruence,

$$\#f^{-1}(z) \cap X(\mathbb{R}) \;\equiv\; d \mod 2\,, \tag{2.1}$$

which holds as the group $\mathbb{Z}_2$ generated by complex conjugation acts freely on the nonreal points of the fiber. When the map f is finite, this congruence (2.1) holds for all z, when multiplicities are taken into account.

There is an additional congruence on the number of real points when there is an additional involution which satisfies a simple hypothesis.

Lemma 6. *Suppose that X has an involution $\angle \curvearrowright X$ written $x \mapsto x^\angle$ satisfying $f(x^\angle) = f(x)$ such that the image $f(X_\angle)$ in Z of the fixed points $X_\angle$ has codimension at least 2, $\dim f(X_\angle) + 2 \leq \dim Z$.*

If $y, z \in Z(\mathbb{R})$ lie in the same connected component of $Z(\mathbb{R})$ and the fibers above them are finite, then

$$\#f^{-1}(y) \cap X(\mathbb{R}) \;\equiv\; \#f^{-1}(z) \cap X(\mathbb{R}) \mod 4\,,$$

where the points are counted with multiplicity.

Proof. We may assume that y and z are regular values of f, as the result for fibers of critical values follows by degeneration to critical fibers. By our assumption on $\dim X_\angle$, it is possible to connect y with z by a path $\gamma\colon [0,1] \to Z(\mathbb{R})$ which does not meet the set $f(X_\angle)(\mathbb{R})$ of points $w \in Z(\mathbb{R})$ whose fiber $f^{-1}(w)$ meets $X_\angle$. Furthermore, we may assume that the image of γ contains at most finitely many critical values of f. Let $S \subset (0,1)$ be those parameter values s for which $\gamma(s)$ is a critical value.

Pulling back regular fibers of f along γ and then taking the closure in $X(\mathbb{C}) \times [0,1]$ gives a map $f_\gamma \colon X_\gamma \to [0,1]$ with finite fibers. If we let c denote complex conjugation, then the group $K := \{e, c, \angle, c\angle\} \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$ acts on the fibers of f_γ. By our assumption on γ and $X_\angle$, there are no $\angle$-fixed points in X_γ.

Thus the points in the fibers of f_γ come in three types;

(1) points of $X(\mathbb{R})$, which are those fixed by c,
(2) points of $X(\mathbb{C}) \setminus X(\mathbb{R})$ on which K acts freely, and
(3) points of $X(\mathbb{C}) \setminus X(\mathbb{R})$ that are fixed by $c\angle$.

The numbers of points of each type is locally constant on $[0,1] \smallsetminus S$, as it may only change at a critical value $s \in S$.

If the number of real points in a fiber changes at s then two real points $r_1(t), r_2(t) \in f_\gamma^{-1}(t)$ come together at a point $r \in f_\gamma^{-1}(s)$ as t approaches s from one direction, and then become a complex conjugate pair as t passes s. As $\angle$ acts fiberwise and has no fixed points, there is another pair of real points $r_1(t)^\angle$ and $r_2(t)^\angle$ in the fiber which approach $r^\angle$ as t approaches s and become a complex conjugate pair as t passes s.

This shows that the number of real points in a fiber can only change by a multiple of four, and completes the proof. ☐

Remark 7. We could have argued symmetrically that the number of points of type (3) may only change in multiples of four. This follows from Lemma 6 as the composition $c\angle$ of the two involutions c and $\angle$ gives a second real structure on X.

Corollary 8 (Of the proof of Lemma 6)**.** *Let $f \colon X \to Z$ and $\angle$ be as above. Then every point of $X_\angle$ is a critical point of f.*

Proof. Given a point $x \in X_\angle$, consider a curve C in Z that contains $f(x)$, is smooth at $f(x)$, and only meets $f(X_\angle)$ at $f(x)$. Then $\angle$ acts fiberwise on the inverse image $f^{-1}(C)$ of C in X. We have that x is a fixed point and $\angle$ acts freely on other points in $f^{-1}(C)$ lying in a neighborhood of x. This implies that at least two branches of C come together at x and proves the corollary. ☐

3. A mod 4 congruence in the real Schubert calculus

Let m be a nonnegative integer. Let us work in a vector space $V \simeq \mathbb{C}^{2m}$ equipped with the ordered basis $\mathbf{e}_0, \mathbf{e}_1, \ldots, \mathbf{e}_{2m-1}$ whose dual space V^* is equipped with dual basis $\mathbf{e}_0^*, \ldots, \mathbf{e}_{2m-1}^*$. That is $(\mathbf{e}_i, \mathbf{e}_j^*) = \delta_{i,j}$ where $(\cdot,\cdot)$ is the pairing between V and V^*. Let $g \colon \mathbb{C} \to V$ be the rational normal curve $g(t) := \sum_i \mathbf{e}_i t^i/i!$. Evaluating an element $u = \sum_i u_i \mathbf{e}_i^*$ of V^* on $g(t)$ gives a polynomial $(g(t), u) = \sum_i u_i t^i/i!$, and this identifies V^* with $\mathbb{C}_{2m-1}[t]$. Under this identification, the canonical symplectic form of Subsection 1.2 becomes $\langle u, v\rangle = uJ^{-1}v$,where $J^{-1} := \sum_i (-1)^i \mathbf{e}_i \otimes \mathbf{e}_{2m-1-i}$ is an isomorphism $V^* \to V$. Its inverse, $J \colon V \to V^*$, is $\sum_i (-1)^i \mathbf{e}_i^* \otimes \mathbf{e}_{2m-1-i}^*$, which gives a symplectic form on V, $\langle p, q\rangle := pJq$ that is dual to the form on V^*.

Let $\eta \in \mathrm{End}(V)$ be the following tensor,

$$\eta \ := \ \sum_{i=0}^{2m-2} \mathbf{e}_{i+1} \otimes \mathbf{e}_i^* \ .$$

This is nilpotent, $\eta^{2m}=0$. Since η has trace zero, it lies in $\mathfrak{sl}_{2m}$, the Lie algebra of the special linear group $\mathrm{SL}(V)$. Moreover, as a simple computation shows that $\eta^T J + J\eta = 0$, it also lies in $\mathfrak{sp}(V)$, the Lie algebra of the symplectic group $\mathrm{Sp}(V)$ (which is defined with respect to the form $\langle\cdot,\cdot\rangle$ on V).

For $t\in\mathbb{C}$, define

$$F_m(t) \ := \ e^{\eta t}.\mathrm{span}\{\mathbf{e}_0,\dots,\mathbf{e}_{m-1}\}\,. \tag{3.1}$$

Since $\mathrm{span}\{\mathbf{e}_0,\dots,\mathbf{e}_{m-1}\}$ is Lagrangian and as $\eta\in\mathfrak{sp}(V)$, we have $e^{\eta t}\in\mathrm{Sp}(V)$, this (3.1) defines a curve in $\mathrm{LG}(m)\subset\mathrm{Gr}(m,V)$. Taking the limit as $t\to\infty$ gives $F_m(\infty)=\mathrm{span}\{\mathbf{e}_m,\dots,\mathbf{e}_{2m-1}\}$, and defines $F_m(t)$ for $t\in\mathbb{P}^1$. For $H\in\mathrm{Gr}(m,V)$ there will be m^2 points t of $\mathbb{P}^1$ (counted with multiplicity) for which H meets $F_m(t)$ nontrivially. (We explain this below.) That is, points t such that $H\in X_{\square}F_\bullet(t)$, where $F_\bullet(t)$ is a flag extending $F_m(t)$. Conversely, given m^2 points $t_1,\dots,t_{m^2}$ of $\mathbb{P}^1$, it is a problem in enumerative geometry to ask how many $H\in\mathrm{Gr}(m,V)$ will meet each linear subspace $F_m(t_1),\dots,F_m(t_{m^2})$ nontrivially. By work of Schubert [21] and of Eisenbud and Harris [1], there will be

$$\#^G_{m,m} \ := \frac{(m^2)!\cdot 0!1!\dots(m-1)!}{m!(m+1)!\cdots(2m-1)!}$$

such planes W, counted with multiplicity. (Schubert determined the degree, and Eisenbud and Harris proved finiteness for this problem.) The point of this discussion is that the flags $F_\bullet(t_i)$ are not in general position.

We may also pose a version of this enumerative problem on the Lagrangian Grassmannian. Given generic points $t_1,\dots,t_{\binom{m+1}{2}}$ of $\mathbb{P}^1$, how many Lagrangian subspaces $W\in\mathrm{LG}(V)$ will meet each of $F_m(t_1),\dots,F_m(t_{\binom{m+1}{2}})$ nontrivially? By results of [11, Cor. 3.6] (for the degree) and [25] (for finiteness) there will be

$$\#^{\angle}_m \ := 2^{\binom{m}{2}}\frac{\binom{m+1}{2}!\cdot 0!1!\dots(m-1)!}{1!3!\cdots(2m-1)!}$$

Lagrangian subspaces, counted with multiplicity. In [25], it was shown that there exists a choice of $t_1,\dots,t_{\binom{m+1}{2}}\in\mathbb{RP}^1$ for which there are finitely many such Lagrangian subspaces, and none of them are real.

The first problem in the Schubert calculus on $\mathrm{Gr}(m,V)$ may be recast in the language of Section 2. Let $H\in\mathrm{Gr}(m,V)$ be a m-dimensional linear subspace of V. Its annihilator, $H^\perp\subset V^*$, also has dimension m. Let $f_1,\dots,f_m$ be a basis for $H^\perp$, which we consider to be polynomials in $\mathbb{C}_{2m-1}[t]$. Their Wronskian is the determinant

$$\mathrm{Wr}(f_1,\dots,f_m) \ := \ \det\begin{pmatrix} f_1(t) & f_2(t) & \cdots & f_m(t)\\ f_1'(t) & f_2'(t) & \cdots & f_m'(t)\\ \vdots & \vdots & \ddots & \vdots\\ f_1^{(m-1)}(t) & f_2^{(m-1)}(t) & \cdots & f_m^{(m-1)}(t)\end{pmatrix},$$

which is a polynomial of degree at most m^2. This Wronskian depends on the subspace H up to multiplication by nonzero scalars (coming from different bases for $H^\perp$), and thus gives a well-defined element of the projective space $\mathbb{P}(\mathbb{C}_{m^2}[t])\simeq\mathbb{P}^{m^2}$.

This defines the Wronski map

$$\mathrm{Wr}\ :\ \mathrm{Gr}(m,V)\ \longrightarrow\!\!\!\!\rightarrow\ \mathbb{P}(\mathbb{C}_{m^2}[t])\,. \tag{3.2}$$

To relate the Wronski map to the Schubert calculus in $\mathrm{Gr}(m,V)$ first note that the rational normal curve $g(t)$ is equal to $e^{\eta t}.\mathbf{e}_0$. Moreover, given a point $u=\sum_i u_i\mathbf{e}_i^*\in V^*$, its evaluation, $(e^{\eta t}.\mathbf{e}_i,u)$, on $e^{\eta t}.\mathbf{e}_i$ is the ith derivative of the polynomial $(e^{\eta t}.\mathbf{e}_0,u)=(g(t),u)$ corresponding to u. As above, let $H\in\mathrm{Gr}(m,V)$ and suppose that $f_1,\dots,f_m\in\mathbb{C}_{2m-1}[t]$ are a basis for $H^\perp$. As shown in [26, p. 123], the zeroes of the Wronskian $\mathrm{Wr}(f_1,\dots,f_m)$ are exactly those numbers s where $H\cap F_m(s)\neq\{0\}$.

We restate and prove Theorem 1.

Theorem 1. *Suppose that $m\geq 3$. If Φ is any real polynomial of degree at most m^2, then the number of real subspaces in $\mathrm{Gr}(m,\mathbb{R}_{2m-1}[t])$ whose Wronskian is proportional to Φ is congruent modulo four to $\#^G_{m,m}$, counted with multiplicity.*

Equivalently, given any subset $T=\{t_1,\dots,t_{m^2}\}$ of $\mathbb{P}^1$ which is stable under complex conjugation, $\overline{T}=T$, the number of real subspaces $H\in\mathrm{Gr}(m,V)$ whose complexifications meet each of $F_m(t_1),\dots,F_m(t_{m^2})$ nontrivially is congruent to $\#^G_{m,m}$ modulo four.

We state the key lemma of this paper.

Lemma 9. *The Wronski map commutes with the Lagrangian involution on $\mathrm{Gr}(m,V)$; for $H\in\mathrm{Gr}(m,V)$,*

$$\mathrm{Wr}(H^\angle)\ =\ \mathrm{Wr}(H)\,.$$

Proof of Theorem 1. The Wronski map (3.2) is a proper and dominant map from $\mathrm{Gr}(m,V)$ to $\mathbb{P}(\mathbb{C}_{m^2}[t])$, both of which are irreducible smooth varieties. (This implies that it is finite.) Since $\mathrm{LG}(V)=\mathrm{Gr}(m,V)_\angle$ and $\dim\mathrm{LG}(V)=\frac{1}{2}(m^2+m)$, we see that the image $\mathrm{Wr}(\mathrm{LG}(V))$ in $\mathbb{P}(C_{m^2}[t])$ has codimension at least two when $m\geq 3$. By Lemma 9 the involution $\angle$ on $\mathrm{Gr}(m,V)$ commutes with Wr. Since the set of real points of $\mathbb{P}(\mathbb{C}_{m^2}[t])$ (which is $\mathbb{P}(\mathbb{R}_{m^2}[t])$) are connected, Lemma 6 implies that the number of real points in any two fibers above real polynomials are congruent modulo four. The congruence to $\#^G_{m,m}$ follows as there is a real polynomial $\Phi(t)$ with $\#^G_{m,m}$ real points in $\mathrm{Wr}^{-1}(\Phi)$ [23]. □

Proof of Lemma 9. By continuity, it suffices to show this for H whose Wronskian has only simple roots, as this set is dense in $\mathrm{Gr}(m,V)$. Since $\eta\in\mathfrak{sp}(V)$, we have $e^{\eta t}\in\mathrm{Sp}(V)$. As $F_m(t)=e^{\eta t}.F_m(0)$ and $F_m(0)$ is Lagrangian, we see that if $H\in\mathrm{Gr}(m,V)$ meets $F_m(t)$ nontrivially, then so does $H^\angle$. This implies that $\mathrm{Wr}(H)=\mathrm{Wr}(H^\angle)$. □

4. A congruence modulo four in the symmetric Schubert calculus

We retain the notation and definitions of Section 3 and extend Theorem 1 to more general Schubert problems.

We defined the Lagrangian involution $\angle$ in Subsection 1.3. For a linear subspace K of V, $K^\angle$ is its annihilator with respect to the form $\langle\cdot,\cdot\rangle$,

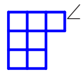
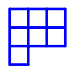
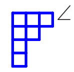
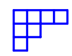
$$K^\angle\ :=\ \{v\in V\mid\langle v,K\rangle\equiv 0\}\,.$$

Since $K \subset L \Leftrightarrow L^{\angle} \subset K^{\angle}$ and $\dim K^{\angle} = 2m - \dim K$, we see that if $F_\bullet$ is a flag, then

$$F_\bullet^{\angle}\ :\ (F_{2m-1})^{\angle}\ \subset\ (F_{2m-2})^{\angle}\ \subset\ \cdots\ \subset (F_2)^{\angle}\ \subset\ (F_1)^{\angle}\ \subset\ V$$

is also a flag. Furthermore, $F_\bullet = F_\bullet^{\angle}$ if and only if $F_\bullet$ is isotropic.

For a partition λ, its *transpose*, $\lambda^{\angle}$, is given by the matrix transpose of Young diagrams,

 =  and  =  .

A partition λ is symmetric if it equals it transpose, $\lambda = \lambda^{\angle}$.

Lemma 10. *Let $\lambda\colon m \geq \lambda_1 \geq \cdots \geq \lambda_m \geq 0$ be a partition and $F_\bullet$ a flag. Then*

$$\angle(X_\lambda F_\bullet)\ :=\ \{H^{\angle} \mid H \in X_\lambda F_\bullet\}\ =\ X_{\lambda^{\angle}} F_\bullet^{\angle}\,.$$

Proof. Write $\binom{[2m]}{m}$ for the set of all subsets of $[2m]$ of cardinality m. To a partition λ, we associate an element of $\binom{[2m]}{m}$,

$$a(\lambda)\ :=\ \{m+1-\lambda_1, m+2-\lambda_2, \ldots, m+m-\lambda_m\}\,.$$

We define an involution on $\binom{[2m]}{m}$,

$$\binom{[2m]}{m} \ni \alpha\ \longmapsto\ \alpha^{\angle}\ :=\ \{2m+1-i \mid i \notin \alpha\}\,.$$

It is a pleasing combinatorial exercise to show that

$$a(\lambda)^{\angle}\ =\ a(\lambda^{\angle})\,.$$

To prove the lemma, we use the following reformulation of the definition (1.1) of a Schubert variety, which we leave as an exercise,

$$X_\lambda F_\bullet\ =\ \{H \in \mathrm{Gr}(m,V) \mid \dim H \cap F_i \geq \#(a(\lambda) \cap [i]), \quad i = 1, \ldots, 2m\}\,.$$

We have the following chain of equivalences for $\alpha \in \binom{[2m]}{m}$ and flags $F_\bullet$.

$$\begin{aligned}
&\dim H \cap F_i\ \geq\ \#(\alpha \cap [i])\\
&\qquad\Leftrightarrow\ \dim(\mathrm{span}\{H, F_i\})\ \leq\ m + i - \#(\alpha \cap [i])\ =\ i + \#(\alpha \cap \{i{+}1, \ldots, 2m\})\\
&\qquad\Leftrightarrow\ \dim(\mathrm{span}\{H, F_i\})^{\angle}\ \geq 2m - i - \#(\alpha \cap \{i{+}1, \ldots, 2m\})\,.
\end{aligned}$$

But we have $(\mathrm{span}\{H, F_i\})^{\angle} = H^{\angle} \cap (F_i)^{\angle} = H^{\angle} \cap F^{\angle}_{2m-i}$ and

$$2m - i - \#(\alpha \cap \{i{+}1, \ldots, 2m\})\ =\ \#(\alpha^{\angle} \cap [2m{-}i])\,,$$

which completes the proof. $\square$

Corollary 11. *Let $F_\bullet$ be a isotropic flag and λ a partition. Then $\angle(X_\lambda F_\bullet) = X_\lambda F_\bullet$ if and only if λ is symmetric. When λ is symmetric, the set of fixed points $(X_\lambda F_\bullet)_\angle$ of $X_\lambda F_\bullet$ is $Y_\lambda F_\bullet$.*

In Section 3 we defined a family $F_m(t)$ for $t \in \mathbb{P}^1$ of Lagrangian subspaces of V. We extend this to a family of isotropic flags. For $j = 1, \ldots, 2m$ and $t \in \mathbb{C}$ define

$$F_j \ := \ \mathrm{span}\{\mathbf{e}_0, \ldots, \mathbf{e}_{j-1}\} \qquad \text{and} \qquad F_j(t) \ := \ e^{\eta t}.F_j\,.$$

As $\langle F_j, F_{2m-j}\rangle = 0$, the subspaces F_j define an isotropic flag, $F_\bullet$. Since $e^{\eta t} \in \mathrm{Sp}(V)$, the subspaces $F_j(t)$ also form an isotropic flag $F_\bullet(t)$ with $F_\bullet(0) = F_\bullet$. Taking the limit as $t \to \infty$ gives

$$F_j(\infty) \ = \ \mathrm{span}\{\mathbf{e}_{2m-j}, \ldots, \mathbf{e}_{2m-1}\}\,,$$

and so we also have that $F_\bullet(\infty)$ is isotropic. This family of flags has the property that if $a \neq b$ are points of $\mathbb{P}^1$, then $F_\bullet(a)$ is in linear general position with respect to $F_\bullet(b)$.

Let $\boldsymbol{\lambda} = (\lambda^1, \ldots, \lambda^n)$ be a Schubert problem and consider a family of all Schubert problems given by flags $F_\bullet(t)$ for $t \in \mathbb{P}^1$. Write $(\mathbb{P}^1)^n_{\neq}$ for the set of n-tuples of distinct points. Let $X_{\boldsymbol{\lambda}} \subset \mathrm{Gr}(m, V) \times (\mathbb{P}^1)^n$ be the closure of the incidence variety

$$X^\circ_{\boldsymbol{\lambda}} \ := \ \{(H, t_1, \ldots, t_n) \mid (t_1, \ldots, t_n) \in (\mathbb{P}^1)^n_{\neq} \text{ and } H \in X_{\lambda^i} F_\bullet(t_i)\ i = 1, \ldots, n\}\,.$$

Let $f\colon X_{\boldsymbol{\lambda}} \to (\mathbb{P}^1)^n$ be the projection. Since for $\boldsymbol{t} = (t_1, \ldots, t_n) \in (\mathbb{P}^1)^n_{\neq}$, we have

$$f^{-1}(\boldsymbol{t}) \ = \ X_{\lambda^1} F_\bullet(t_1) \cap X_{\lambda^2} F_\bullet(t_2) \cap \cdots \cap X_{\lambda^n} F_\bullet(t_n)\,, \tag{4.1}$$

this family $f\colon X_{\boldsymbol{\lambda}} \to (\mathbb{P}^1)^n$ contains all instances of the Schubert problem $\boldsymbol{\lambda}$ given by flags $F_\bullet(t)$.

Lemma 12. *For any Schubert problem $\boldsymbol{\lambda}$ on $\mathrm{Gr}(m, V)$, the map $f\colon X_{\boldsymbol{\lambda}} \to (\mathbb{P}^1)^n$ is finite and has degree $d(\boldsymbol{\lambda})$.*

Remark 13. Speyer [27] constructed and studied a more refined compactification of $X^\circ_{\boldsymbol{\lambda}}$. Since $\mathrm{PGL}(2, \mathbb{C})$ acts on $\mathbb{P}^1$, and through the one-parameter subgroup $e^{\eta t}$ it acts on V and on $\mathrm{Gr}(m, V)$, we have that $\mathrm{PGL}(2, \mathbb{C})$ acts on the family $X^\circ_{\boldsymbol{\lambda}} \to (\mathbb{P}^1)^n_{\neq}$. The orbit space is a family of Schubert problems over $M_{0,n} := (\mathbb{P}^1)^n_{\neq}/\mathrm{PGL}(2, \mathbb{C})$, the open moduli space of n marked points on $\mathbb{P}^1$, which Speyer extended to a family over its compactification $\overline{M}_{0,n}$.

Proof of Lemma 12. Let us assume that $d(\boldsymbol{\lambda}) \neq 0$ for otherwise $X_{\boldsymbol{\lambda}} = \emptyset$. Consider the map $\chi\colon (\mathbb{P}^1)^n \to \mathbb{P}(\mathbb{C}_{m^2}[t])$ given by

$$\chi \ : \ (t_1, \ldots, t_n) \ \longmapsto \ (t - t_1)^{|\lambda^1|}(t - t_2)^{|\lambda^2|} \cdots (t - t_n)^{|\lambda^n|}\,,$$

and let $X_* \to (\mathbb{P}^1)^n$ be the pullback of the Wronski map

$$\mathrm{Wr} \ : \ \mathrm{Gr}(m, V) \ \longrightarrow \ \mathbb{P}(\mathbb{C}_{n^2}[t])$$

along the map χ.

Since the Wronski map is finite (it is a surjective map of smooth complete varieties with finite fibers), the map $X_* \to (\mathbb{P}^1)^n$ is finite. Purbhoo studied the fibers of the Wronski map [20]. The set-theoretic fiber over a polynomial $g \in \mathbb{P}(\mathbb{C}_{n^2}[t])$ is

$$\bigcap_{\{s\colon g(s)=0\}} \ \bigcup_{\{\nu\colon |\nu| = \mathrm{ord}_s\, g\}} X_\nu F_\bullet(s)\,.$$

In the scheme-theoretic fiber, the Schubert variety $X_\nu F_\bullet(s)$ in this intersection is not reduced; it has multiplicity equal to the number of standard Young tableaux of shape ν. Thus the set-theoretic fiber of X_* over a point $(t_1, \ldots, t_n) \in (\mathbb{P}^1)^n_{\neq}$ is

$$\bigcap_{i=1}^{n} \bigcup_{\{\nu\colon |\nu|=|\lambda^i|\}} X_\nu F_\bullet(t_i)\,.$$

It follows that $X_{\boldsymbol{\lambda}}$ is a union of irreducible components of X_* and so $X_{\boldsymbol{\lambda}} \to (\mathbb{P}^1)^n$ is finite. $\square$

A Schubert problem $\boldsymbol{\lambda} = (\lambda^1, \ldots, \lambda^n)$ is *symmetric* if each partition λ^i is symmetric. By Lemma 10, if $\boldsymbol{\lambda}$ is a symmetric Schubert problem, then the fibers of $X_{\boldsymbol{\lambda}} \to (\mathbb{P}^1)^n$ are preserved by the Lagrangian involution. We would like to establish an analog of Theorem 1 for symmetric Schubert problems, that the number of real points in a fiber is congruent to $d(\boldsymbol{\lambda})$ modulo four. Unfortunately, $X_{\boldsymbol{\lambda}} \to (\mathbb{P}^1)^n$ is not the right family for this. If $(t_1, \ldots, t_n)$ is a real point in $(\mathbb{P}^1)^n_{\neq}$, then each t_i is real and the Mukhin-Tarasov-Varchenko Theorem [15, 16] states that the intersection (4.1) consists of $d(\boldsymbol{\lambda})$ real points, so that the desired congruence to $d(\boldsymbol{\lambda})$ modulo four is not interesting.

The problem is that the points $t_1, \ldots, t_n$ are distinguishable. Since $\overline{F_i(t)} = F_i(\overline{t})$, we have

$$\overline{X_\lambda F_\bullet(t)} \;:=\; \{\overline{H} \mid H \in X_\lambda F_\bullet(t)\} \;=\; X_\lambda F_\bullet(\overline{t})\,, \tag{4.2}$$

and so there may be fibers of $X_{\boldsymbol{\lambda}} \to (\mathbb{P}^1)^n$ which are real (stable under complex conjugation), for which the corresponding ordered n-tuple $(t_1, \ldots, t_n)$ is not fixed by complex conjugation. To remedy this, we replace $(\mathbb{P}^1)^n$ by a space in which points t_i and t_j are indistinguishable when $\lambda^i = \lambda^j$.

This uses an alternative representation of a symmetric Schubert problem which records the frequency of the different partitions. Suppose that $\{\mu^1, \ldots, \mu^r\}$ are the distinct partitions appearing in a Schubert problem $\boldsymbol{\lambda}$ so that $\mu^i \neq \mu^j$ for $i \neq j$, and μ^i occurs $n_i > 0$ times in $\boldsymbol{\lambda}$. In that case, we write $[\boldsymbol{\mu}] = \{(\mu^1, n_1), \ldots, (\mu^r, n_r)\}$ for the Schubert problem. For example,

$$\bigl\{\bigl(\ydiagram{2,2}, 2\bigr)\,,\ \bigl(\ydiagram{2,1}, 4\bigr)\,,\ \bigl(\ydiagram{1}, 5\bigr)\bigr\} \qquad \text{and} \qquad \bigl\{\bigl(\ydiagram{3,2,1}, 1\bigr)\,,\ \bigl(\ydiagram{2,1}, 5\bigr)\,,\ \bigl(\ydiagram{1}, 4\bigr)\bigr\}$$

are symmetric Schubert problems on $\mathrm{Gr}(5, \mathbb{C}^{10})$. We may write $[\boldsymbol{\mu}] = [\boldsymbol{\mu}](\boldsymbol{\lambda})$ to indicate that $[\boldsymbol{\mu}]$ and $\boldsymbol{\lambda}$ are the same Schubert problem and define $d([\boldsymbol{\mu}]) := d(\boldsymbol{\lambda})$ when this occurs.

Given a symmetric Schubert problem in this form $[\boldsymbol{\mu}] = \{(\mu^1, n_1), \ldots, \{\mu^r, n_r)\}$, set

$$Z_{[\boldsymbol{\mu}]} \;:=\; \{(u_1, u_2, \ldots, u_r) \mid u_i \in \mathbb{P}(\mathbb{C}_{n_i}[t])\} \;\simeq\; \prod_{i=1}^{r} \mathbb{P}^{n_i}\,,$$

and let $U_{[\boldsymbol{\mu}]}$ be those $(u_1, \ldots, u_r) \in Z_{[\boldsymbol{\mu}]}$ where each polynomial u_i is square-free and any two are relatively prime (i.e. the roots of $u_1 \cdots u_r$ all have multiplicity one). Let $X_{[\boldsymbol{\mu}]} \subset \mathrm{Gr}(m, V) \times Z_{[\boldsymbol{\mu}]}$ be the closure of the incidence correspondence

$$\{(H\,,\, u_1, \ldots, u_r) \mid (u_1, \ldots, u_r) \in U_{[\boldsymbol{\mu}]} \text{ and } H \in X_{\mu^i}(s) \text{ for } u_i(s) = 0\,, i = 1, \ldots, r\}\,.$$

Let $f\colon X_{[\boldsymbol{\mu}]} \to Z_{[\boldsymbol{\mu}]}$ be the projection. Since for $\boldsymbol{u} = (u_1, \ldots, u_r) \in U_{[\boldsymbol{\mu}]}$, we have

$$f^{-1}(\boldsymbol{u}) \;=\; \bigcap_{i=1}^{r} \bigcap_{\{s\colon u_i(s)=0\}} X_{\mu^i} F_\bullet(s)\,, \tag{4.3}$$

the family $f\colon X_{[\boldsymbol{\mu}]} \to U_{[\boldsymbol{\mu}]}$ is another family containing all instances of the Schubert problem $[\boldsymbol{\mu}]$ given by flags $F_\bullet(t)$.

By (4.2) an intersection (4.3) is stable under complex conjugation exactly when the roots of u_i are stable under complex conjugation. That is, exactly when each u_i is a real polynomial, so that $(u_1, \ldots, u_r) \in U_{[\boldsymbol{\mu}]}(\mathbb{R})$.

Lemma 14. *The map $f\colon X_{[\boldsymbol{\mu}]} \to Z_{[\boldsymbol{\mu}]}$ is finite.*

This has nearly the same proof as does the finiteness of $X_{\boldsymbol{\lambda}} \to (\mathbb{P}^1)^n$. We just replace the map χ by the map $\varphi\colon Z_{[\boldsymbol{\mu}]} \to \mathbb{P}(\mathbb{C}_{n^2}[t])$ defined by

$$Z_{[\boldsymbol{\mu}]} \;\ni\; (u_1, \ldots, u_r) \;\longmapsto\; u_1^{|\mu^1|} \cdot u_2^{|\mu^2|} \cdots u_r^{|\mu^r|} \;\in\; \mathbb{P}(\mathbb{C}_{m^2}[t])\,.$$

When $[\boldsymbol{\mu}] = [\boldsymbol{\mu}](\boldsymbol{\lambda})$ so that $[\boldsymbol{\mu}]$ and $\boldsymbol{\lambda}$ are the same Schubert problem, the map $\chi\colon (\mathbb{P}^1)^n \to \mathbb{P}(\mathbb{C}_{m^2}[t])$ factors through φ. Define the map $\psi\colon (\mathbb{P}^1)^n \to Z_{[\boldsymbol{\mu}]}$ by

$$\psi \;:\; (t_1, \ldots, t_n) \;\longmapsto\; \Bigl(\prod_{\{j\in[n]\mid\lambda^j=\mu^i\}} (t - t_j) \mid i = 1, \ldots, r\Bigr)\,. \tag{4.4}$$

Then $\chi = \varphi \circ \psi$, and we have the commutative diagram

$$\begin{tikzcd}
X_{\boldsymbol{\lambda}} \arrow[r] \arrow[d, "f"'] & X_{[\boldsymbol{\mu}]} \arrow[r] \arrow[d, "f"'] & \mathrm{Gr}(m, V) \arrow[d, "\mathrm{Wr}"] \\
(\mathbb{P}^1)^n \arrow[r, "\psi"] & Z_{[\boldsymbol{\mu}]} \arrow[r, "\phi"] & \mathbb{P}(\mathbb{C}_{m^2}[t])
\end{tikzcd}$$

The Lagrangian involution $\angle$ on $\mathrm{Gr}(m, V)$ induces an involution $\angle$ on $\mathrm{Gr}(m, V) \times Z_{[\boldsymbol{\mu}]}$ which acts trivially on the second factor. When $[\boldsymbol{\mu}]$ is symmetric, this restricts to an involution $\angle$ on $X_{[\boldsymbol{\mu}]}$ with $f(H^\angle) = f(H)$, by Corollary 11. In this context, the Theorem of Mukhin, Tarasov, and Varchenko implies that if $\boldsymbol{u} = (u_1, \ldots, u_r)$ is a point of $U_{[\boldsymbol{\mu}]}$ in which the polynomials u_i have distinct real roots, then all points in the fiber $f^{-1}(\boldsymbol{u})$ consists of $d(\boldsymbol{\lambda})$ real points. Thus we have the following corollary of Lemma 6 for $f\colon X_{[\boldsymbol{\mu}]} \to Z_{[\boldsymbol{\mu}]}$.

Theorem 15. *Suppose that $[\boldsymbol{\mu}]$ is a symmetric Schubert problem. If the image $f((X_{[\boldsymbol{\mu}]})_\angle)$ in $Z_{[\boldsymbol{\mu}]}$ of the points $(X_{[\boldsymbol{\mu}]})_\angle$ fixed by $\angle$ has codimension at least 2 in $Z_{[\boldsymbol{\mu}]}$, then the number of real points in a fiber over $u \in U_{[\boldsymbol{\mu}]}(\mathbb{R})$ is congruent to $d(\boldsymbol{\lambda})$ modulo four.*

Since the map $\psi\colon (\mathbb{P}^1)^n \to Z_{[\boldsymbol{\mu}]}$ is finite, the map $X_{\boldsymbol{\lambda}} \to X_{[\boldsymbol{\mu}]}$ is finite and so this condition on codimension is equivalent to the image of $(X_{\boldsymbol{\lambda}})_\angle$ in $(\mathbb{P}^1)^n$ having dimension at most $n-2$.

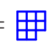
Both aspects of Theorem 15, the condition on codimension and the congruence modulo four, are quite subtle. Below, we will give several examples which serve to illustrate this subtlety. Conjecture 22 below gives a combinatorial condition which should imply the condition on codimension and hence the congruence modulo four, but we cannot prove it as we lack a Bertini-type theorem. We instead offer a weaker condition that we can prove. Recall that if λ is a symmetric partition, then $\ell(\lambda)$ is the number of boxes in the main diagonal of the Young diagram of λ.

Theorem 16. *Let $\boldsymbol{\lambda} = (\lambda, \mu, \nu^1, \dots, \nu^n)$ be a symmetric Schubert problem in which either $\lambda \neq \mu$ or else $\lambda = \mu$ and there is some i with $\lambda = \mu = \nu^i$, and set $[\boldsymbol{\mu}] := [\boldsymbol{\mu}](\boldsymbol{\lambda})$. If*

$$2 \;+\; \frac{1}{2}\left(|\nu^1| + \cdots + |\nu^n| \;+\; m - \ell(\lambda) - \ell(\mu)\right) \;\leq\; n\,, \tag{4.5}$$

then the number of real points in a fiber of $X_{[\boldsymbol{\mu}]}$ over a real point of $U_{[\boldsymbol{\mu}]}$ is congruent to $d(\boldsymbol{\lambda})$ modulo four.

Example 17. The work Eremenko and Gabrielov on lower bounds in the Schubert calculus [2] was generalized in Theorem 6.4 of [22], which applies to the Wronski map restricted to certain intersections of two Schubert varieties. In some cases Theorem 16 applies and the degree is zero but the congruence modulo four implies that every fiber has at least two real points as in Corollary 2.

Let λ and μ be partitions encoding Schubert conditions on $\mathrm{Gr}(m, V)$. These define a *skew partition*, λ^c/μ. Here, λ^c is the partition complementary to λ in that $\lambda^c_i = m - \lambda_{m+1-i}$ for $i = 1, \dots, m$ and λ^c/μ is the collection of boxes that remain in λ^c after removing μ. Implicit in this construction is that $\mu \subset \lambda^c$. Below are two examples, one when $m = 4$ with $\lambda =$  and $\mu =$ , and the other when $m = 5$ with $\lambda =$  and $\mu =$ . The skew partition is unshaded.

(4.6) 

Let $|\lambda^c/\mu| := m^2 - |\lambda| - |\mu|$ be the number of boxes in the skew partition λ^c/μ. Restricting the Wronski map to the intersection $\Omega_{\lambda,\mu} := \Omega_\lambda F_\bullet(\infty) \cap \Omega_\mu F_\bullet(0)$ of Schubert varieties (and dividing by $t^{|\mu|}$) gives a finite map

$$\mathrm{Wr}_{\lambda,\mu} \;:\; \Omega_{\lambda,\mu} \;\longrightarrow\; \mathbb{P}(\mathbb{C}_{|\lambda^c/\mu|}[t])\,. \tag{4.7}$$

Its degree is the number $f(\lambda^c/\mu)$ of standard Young tableaux of skew shape λ^c/μ [6]. The boxes in λ^c/μ form a poset in which a box is covered by it neighbors immediately to its right or below and $f(\lambda^c/\mu)$ is the number of linear extensions of this poset.

The variety $\Omega_{\lambda,\mu}$ in the Plücker embedding of the Grassmannian $\mathrm{Gr}(m, V)$ is a variety whose coordinate ring is an algebra with straightening law on the poset λ^c/μ, which is a distributive lattice. In these Plücker coordinates the map $\mathrm{Wr}_{\lambda,\mu}$ is what is called in [22, §6] a *Wronski projection for λ^c/μ with constant sign*. Furthermore, the subset of $\Omega_{\lambda,\mu}$ where the minimally indexed Plücker coordinate x_μ does not vanish is an open subset of affine space and is therefore orientable. By Theorem 6.4 of [22] the degree (there called characteristic) of the Wronski map is the *sign-imbalance of the poset λ^c/μ*.

This sign-imbalance is computed as follows. Fixing one linear extension of λ^c/μ, all others are obtained from it by a permutation. The sign-imbalance of λ^c/μ is the absolute value of the sum of the signs of these permutations.

When λ and μ are symmetric, transposition induces an involution $T \mapsto T^{\angle}$ on the set of Young tableaux/linear extensions. The permutations corresponding to T and to $T^{\angle}$ differ by the product of transpositions, one for each pair of boxes in λ^c/μ that are interchanged by transposing. Thus when there is an odd number of such pairs, T and $T^{\angle}$ contribute opposite signs to the sign-imbalance. Since $T \neq T^{\angle}$ when $|\lambda^c/\mu| > 1$, this argument also shows that $f(\lambda^c/\mu)$ is even. We deduce the following lemma.

Lemma 18. *When a symmetric skew partition λ^c/μ has an odd number of boxes above its main diagonal, its sign-imbalance is zero.*

The skew partitions in (4.6) have three and five boxes above their main diagonal, respectively, and so both have sign-imbalance zero. For symmetric partitions λ and μ, we have a symmetric Schubert problem $[\boldsymbol{\mu}] := \{(\lambda, 1), (\mu, 1), (\yng(1), |\lambda^c/\mu|)\}$. Then $Z_{[\boldsymbol{\mu}]}$ is the product $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}(\mathbb{C}_{|\lambda^c/\mu|}[t])$ and the restricted Wronski map (4.7) is simply the restriction of the map $X_{[\boldsymbol{\mu}]} \to Z_{[\boldsymbol{\mu}]}$ to the set $\{(0, \infty)\} \times \mathbb{P}(\mathbb{C}_{|\lambda^c/\mu|}[t])$, and the whole family $X_{[\boldsymbol{\mu}]} \to Z_{[\boldsymbol{\mu}]}$ is the closure of the $\mathrm{PGL}(2, \mathbb{C})$-orbit of (4.7). Thus we lose nothing by considering the restricted Wronski map (4.7) in place of $X_{[\boldsymbol{\mu}]} \to Z_{[\boldsymbol{\mu}]}$.

Corollary 19. *Let λ and μ be symmetric partitions. If $4 + m \leq |\lambda^c/\mu| + \ell(\lambda) + \ell(\mu)$, λ^c/μ has an odd number of boxes above its main diagonal, and $f(\lambda^c/\mu)$ is congruent to two modulo four, then the degree of the real restricted Wronski map* (4.7) *is zero, but its fiber over every real polynomial contains at least two real points.*

The condition $4 + m \leq |\lambda^c/\mu| + \ell(\lambda) + \ell(\mu)$ is the condition of Theorem 16 for the Schubert problem $\{(\lambda, 1), (\mu, 1), (\yng(1), |\lambda^c/\mu|)\}$ as $n = |\lambda^c/\mu|$. Thus Corollary 19 gives a class of geometric problems in which the lower bound given by the degree is not sharp. It generalizes Corollary 2 as the problem $\{(\yng(1), 9\}$ with 42 solutions is the only problem on $\mathrm{Gr}(3, 6)$ satisfying the hypotheses of Corollary 19. There are two such Schubert problems on $\mathrm{Gr}(4, 8)$,

$$\{(\yng(3,3,2), 1), (\yng(1), 8)\} \text{ with 90 solutions} \quad \text{and} \quad \{(\yng(3,1,1), 1), (\yng(2,2), 1), (\yng(1), 8)\} \text{ with 426 solutions.}$$

We give the numbers of problems coming from Corollary 19 for small values of m.

m	3	4	5	6	7
number	1	2	7	18	34

The problem for the second skew tableau of (4.6) has $40,370$ solutions, and the largest on $\mathrm{Gr}(7, 14)$ has $\lambda = (5, 4, 2, 2, 1)$, $\mu = (6, 5, 2, 2, 2, 1)$, and $843,201,530$ solutions. □

Example 20. Consider the Schubert problem $\boldsymbol{\lambda} = (\yng(3,2,1), \yng(3,1,1), \yng(1), \yng(1), \yng(1), \yng(1), \yng(1))$ on $\mathrm{Gr}(4, 8)$ with 40 solutions. Here $n = 5$, $\lambda = \yng(3,2,1)$, $\mu = \yng(3,1,1)$, and $\nu^1, \ldots, \nu^5 = \yng(1)$. Then (4.5) is

$$5 \ \geq \ 2 + \frac{1}{2}(1 + 1 + 1 + 1 + 1 \ + \ 4 - 2 - 1) \ = \ 5\,.$$

So by Theorem 16, there will be a congruence modulo four, which we have observed. Table 2 displays the result of a computation which took 30.6 gigaHertz-days. The columns are labeled by the possible numbers of real solutions, and each cell records how many computed instances had that number of real solutions. □

Table 2. Schubert problem $\{(\square, 5)\,,\ (\ydiagram{3,1,1}, 1)\,,\ (\ydiagram{3,2,1}, 1)\}$ with 40 solutions.

Num. real	0	2	4	6	8	10	12	14	16	18	20
Frequency	181431	0	62673	0	48005	0	29422	0	8360	0	11506

Num. real	22	24	26	28	30	32	34	36	38	40	Total
Frequency	0	14137	0	6123	0	2040	0	9696	0	226607	600000

Proof of Theorem 16. Let $\boldsymbol{\lambda} = (\lambda, \mu, \nu^1, \ldots, \nu^n)$ be a symmetric Schubert problem for $\mathrm{Gr}(m, V)$. Recall that $f\colon X^\circ_{\boldsymbol{\lambda}} \to (\mathbb{P}^1)^{2+n}_{\neq}$ is the family whose fiber over $(a, b, t_1, \ldots, t_n) \in (\mathbb{P}^1)^{2+n}_{\neq}$ is

$$X_\lambda F_\bullet(a) \,\cap\, X_\mu F_\bullet(b) \,\cap\, \bigcap_{i=1}^{n} X_{\nu^i} F_\bullet(t_i)\,.$$

Consider its closure $\widetilde{X}_{\boldsymbol{\lambda}}$ in $\mathrm{Gr}(m, V) \times (\mathbb{P}^1)^2_{\neq} \times (\mathbb{P}^1)^n$—the points a and b defining the first two Schubert varieties remain distinct, but any other pair may collide.

The fiber of $\widetilde{X}_{\boldsymbol{\lambda}}$ over a point $\boldsymbol{t} := (a, b, t_1, \ldots, t_n) \in (\mathbb{P}^1)^{2+n}_{\neq}$ is a subset of

$$X_\lambda F_\bullet(a) \,\cap\, X_\mu F_\bullet(b)\,.$$

The fiber of $(\widetilde{X}_{\boldsymbol{\lambda}})_\angle$ over the same point $\boldsymbol{t}$ is a subset of $(X_\lambda F_\bullet(a) \cap X_\mu F_\bullet(b))_\angle$, which is

$$(X_\lambda F_\bullet(a))_\angle \,\cap\, (X_\mu F_\bullet(b))_\angle \;=\; Y_\lambda F_\bullet(a) \,\cap\, Y_\mu F_\bullet(b)\,. \tag{4.8}$$

As $a \neq b$, the flags $F_\bullet(a)$ and $F_\bullet(b)$ are in linear general position and so this intersection is generically transverse in $\mathrm{LG}(V)$. Thus (4.8) has dimension

$$\begin{aligned}
\dim \mathrm{LG}(V) - \|\lambda\| - \|\mu\| \;&=\; \tfrac{1}{2}(m^2 + m) \;-\; \tfrac{1}{2}(|\lambda| + \ell(\lambda)) \;-\; \tfrac{1}{2}(|\mu| + \ell(\mu))\\
&=\; \tfrac{1}{2}(m^2 + m - |\lambda| - |\mu| - \ell(\lambda) - \ell(\mu))\\
&=\; \tfrac{1}{2}(|\nu^1| + \cdots + |\nu^n| \;+\; m - \ell(\lambda) - \ell(\mu))\,, \qquad (4.9)
\end{aligned}$$

because $|\lambda| + |\mu| + |\nu^1| + \cdots + |\nu^n| = m^2$ as $\boldsymbol{\lambda}$ is a Schubert problem on $\mathrm{Gr}(m, V)$.

Let $\mathcal{Y}_{\lambda,\mu} \subset \mathrm{LG}(V) \times (\mathbb{P}^1)^2_{\neq}$ be the family of intersections (4.8)

$$\{(H, a, b) \;\mid\; H \in Y_\lambda F_\bullet(a) \cap Y_\mu F_\bullet(b)\}\,,$$

which has dimension

$$\dim \mathcal{Y}_{\lambda,\mu} \;=\; 2 \;+\; \frac{1}{2}(|\nu^1| + \cdots + |\nu^n| \;+\; m - \ell(\lambda) - \ell(\mu))\,.$$

Forgetting $(t_1, \ldots, t_n)$ gives a map $\pi\colon (\widetilde{X}_{\boldsymbol{\lambda}})_\angle \to \mathcal{Y}_{\lambda,\mu}$. Its fiber over a point (H, a, b) is

$$\{(t_1, \ldots, t_n) \;\mid\; H \in X_{\nu^i} F_\bullet(t_i) \text{ for } i = 1, \ldots, n\}\,.$$

Since $H \in X_\nu F_\bullet(t)$ implies that $H \in X_{\square} F_\bullet(t)$, and this second condition occurs for only finitely many $t \in \mathbb{P}^1$ (these are the zeroes of the Wronskian of H), the fiber $\pi^{-1}(H, a, b)$

is either empty or it is finite. Thus

$$\dim(\widetilde{X}_{\boldsymbol{\lambda}})_{\angle} \;\leq\; 2 \;+\; \frac{1}{2}(|\nu^1|+\cdots+|\nu^n| \;+\; m-\ell(\lambda)-\ell(\mu))\,. \tag{4.10}$$

Thus the condition (4.5) implies that

$$\dim(f((\widetilde{X}_{\boldsymbol{\lambda}})_{\angle})) \;\leq\; \dim(\widetilde{X}_{\boldsymbol{\lambda}})_{\angle} \;\leq\; n \;=\; \dim((\mathbb{P}^1)^2_{\neq}\times(\mathbb{P}^1)^n)-2\,.$$

To complete the proof, let $[\boldsymbol{\mu}] := [\boldsymbol{\mu}](\boldsymbol{\lambda})$ and consider the map ψ defined in (4.4)

$$\psi\;:\;(\mathbb{P}^1)^2_{\neq}\times(\mathbb{P}^1)^n \;\longrightarrow\; Z_{[\boldsymbol{\mu}]}\,.$$

To apply the argument in the proof of Lemma 6 to the family $f\colon X_{[\boldsymbol{\mu}]}\to Z_{[\boldsymbol{\mu}]}$ requires that there exists a curve $\gamma\subset Z_{[\boldsymbol{\mu}]}(\mathbb{R})$ connecting any two points $\boldsymbol{u},\boldsymbol{u}'\in U_{[\boldsymbol{\mu}]}(\mathbb{R})$ such that $f^{-1}(\gamma)\cap(X_{[\boldsymbol{\mu}]})_{\angle}=\emptyset$. This is always possible if the codimension of $f((X_{[\boldsymbol{\mu}]})_{\angle})$ is at least 2, $Z_{[\boldsymbol{\mu}]}$ is smooth, and $Z_{[\boldsymbol{\mu}]}(\mathbb{R})$ is connected.

From the first part of this proof, we have control over the points of $(X_{[\boldsymbol{\mu}]})_{\angle}$ lying over the image of ψ. Thus we seek a curve γ lying in the real points of the image of ψ. This image consists of polynomials $(u_1,\ldots,u_r)\in Z_{[\boldsymbol{\mu}]}$ where the root a corresponding to λ is distinct from the root b corresponding to μ. The curve γ connecting $\boldsymbol{u},\boldsymbol{u}'\in U_{[\boldsymbol{\mu}]}(\mathbb{R})$ must be a curve of real polynomials where these roots remain distinct. This can always be done when either $\lambda\neq\mu$ (so that a and b are roots of different polynomials), or else $\lambda=\mu$ and there is some i with $\mu=\nu^i$. In this second case, we require that the polynomial u_j corresponding to this common partition has at least two distinct roots at every point of γ—which is possible, as $\deg u_j\geq 3$. ∎

Example 21. The condition of Theorem 16 is sufficient, but by no means necessary for there to be a congruence modulo four. In particular, the estimate (4.10) on the dimension of $(\widetilde{X}_{\boldsymbol{\lambda}})_{\angle}$ could be improved. Consider the problem $\boldsymbol{\lambda}=((3,1,1),(2,2),(2,2),(1),(1),(1))$ on $\mathrm{Gr}(4,8)$ with 12 solutions. For Condition (4.5), we have $n=4$, $\lambda=(3,1,1)$, $\mu=(2,2)$, and $\nu^1,\ldots,\nu^4$ equal to $(2,2),(1),(1),(1)$. Then (4.5) becomes

$$4\geq 2+\frac{1}{2}(4+1+1+1\;+\;4-1-2)\;=\;6\,,$$

which does not hold. Nevertheless, we observed a congruence modulo four in this Schubert problem. Table 3 displays the results of a computation that consumed 150.8 gigaHertz-days of computing. While we did not explicitly compute $\dim(f((\widetilde{X}_{\boldsymbol{\lambda}})_{\angle}))$, it is at most 3,

TABLE 3. Schubert problem $\{((1),3)\,,\,((2,2),2)\,,\,((3,1,1),1)\}$ with 12 solutions.

Num. real	0	2	4	6	8	10	12	Total
Frequency	214375	0	231018	0	61600	0	293007	800000

and we expect it to be 2, based on heuristic arguments that we give below. ∎

Suppose that $\boldsymbol{\lambda}$ is symmetric and $\boldsymbol{t}=(t_1,\ldots,t_n)\in(\mathbb{P}^1)^n_{\neq}$. Then the fiber of $X_{\boldsymbol{\lambda}}$ over $\boldsymbol{t}$ is the intersection (4.1). By Corollary 11, the points of $(X_{\boldsymbol{\lambda}})_{\angle}$ lying over $\boldsymbol{t}$ are

$$Y_{\lambda^1}F_\bullet(t_1)\;\cap\;Y_{\lambda^2}F_\bullet(t_2)\;\cap\;\cdots\;\cap\;Y_{\lambda^n}F_\bullet(t_n)\,,$$

which is a subscheme of $\mathrm{LG}(V)$. Since the codimension of $Y_\lambda F_\bullet(t)$ in $\mathrm{LG}(V)$ is $\|\lambda\|$, it is reasonable to conjecture that the expected dimension of such an intersection gives the dimension of the image of $(X_{\boldsymbol{\lambda}})_\angle$ in $(\mathbb{P}^1)^n$, which would then imply a congruence modulo four. We make a conjecture based on these observations.

Conjecture 22. *Suppose that $\boldsymbol{\lambda} = (\lambda^1, \ldots, \lambda^n)$ is a symmetric Schubert problem for $\mathrm{Gr}(m, V)$. If we have*

$$2 \ \leq \ \|\lambda^1\| \ + \ \cdots \ + \ \|\lambda^n\| \ - \ \dim \mathrm{LG}(V)\,, \tag{4.11}$$

then the number of real points in a fiber of $X_{[\boldsymbol{\mu}]}$ over a real point of $U_{[\boldsymbol{\mu}]}$ is congruent to $d(\boldsymbol{\lambda})$ modulo four, where $[\boldsymbol{\mu}] = [\boldsymbol{\mu}](\boldsymbol{\lambda})$.

Lemma 23. *The inequality* (4.5) *implies the inequality* (4.11)*.*

Proof. Let $\boldsymbol{\lambda} = (\lambda, \mu, \nu^1, \ldots, \nu^n)$ be a symmetric Schubert problem on $\mathrm{Gr}(m, V)$. Rewrite the inequality (4.5) as

$$2 \ + \ \frac{1}{2}(m - \ell(\lambda) - \ell(\mu)) \ \leq \ n \ - \ \frac{1}{2}(|\nu^1| + \cdots + |\nu^n|)\,.$$

Since $1 \leq \|\nu\| = \frac{1}{2}(|\nu| + \ell(\nu))$, this implies that

$$2 \ + \ \frac{1}{2}(m - \ell(\lambda) - \ell(\mu)) \ \leq \ \frac{1}{2}(\ell(\nu^1) + \cdots + \ell(\nu^n))\,.$$

As $\boldsymbol{\lambda}$ is a Schubert problem on $\mathrm{Gr}(m, V)$, we have $m^2 = |\lambda| + |\mu| + |\nu^1| + \cdots + |\nu^n|$, and so this becomes

$$\begin{aligned} 2 \ &\leq \ \frac{1}{2}\big(\ell(\lambda) + \ell(\mu) + \ell(\nu^1) + \cdots + \ell(\nu^n) - m + |\lambda| + |\mu| + |\nu^1| + \cdots + |\nu^n| - m^2\big) \\ &= \ \|\lambda\| + \|\mu\| + \|\nu^1\| + \cdots + \|\nu^n\| - \frac{1}{2}(m^2 + m)\,, \end{aligned}$$

which is the inequality (4.11) of Conjecture 22 as $\dim \mathrm{LG}(V) = \frac{1}{2}(m^2 + m)$ ∎

Remark 24. For the Schubert problem of Example 21, we have

$$\begin{aligned} \|\ydiagram{1}\| + \|\ydiagram{1}\| + \|\ydiagram{1}\| + \|\ydiagram{2,2}\| + \|\ydiagram{2,2}\| + \|\ydiagram{3,1,1}\| \ &- \ \dim \mathrm{LG}(\mathbb{C}^8) \\ &= \ 1 + 1 + 1 + 3 + 3 + 3 - 10 \ = \ 2\,. \end{aligned}$$

Thus the inequality (4.11) holds, and so Conjecture 22 predicts the congruence modulo four that we observed in Example 21.

In every example of a Schubert problem we have computed in which the inequality (4.11) holds, we have observed this congruence to $d(\boldsymbol{\lambda})$ modulo four.

The intuition behind Conjecture 22 is the following. Let $\mathcal{Y}_\lambda \subset \mathrm{LG}(V) \times \mathbb{P}^1$ be $\{(H, t) \mid H \in Y_\lambda F_\bullet(t)\}$, which is the family over $\mathbb{P}^1$ whose fiber over $t \in \mathbb{P}^1$ is $Y_\lambda F_\bullet(t)$. This has codimension $\|\lambda\|$ in $\mathrm{LG}(V) \times \mathbb{P}^1$. Let $\Delta\colon \mathrm{LG}(V) \to \mathrm{LG}(V)^n$ be the diagonal map. Suppose that $\boldsymbol{\lambda} = (\lambda^1, \ldots, \lambda^n)$ is a symmetric Schubert problem. Then the fiber product of $\mathcal{Y}_{\lambda^1}, \ldots, \mathcal{Y}_{\lambda^n}$ over $\mathrm{LG}(V)$ is

$$\Big(\mathcal{Y}_{\lambda^1} \times \mathcal{Y}_{\lambda^2} \times \cdots \times \mathcal{Y}_{\lambda^n}\Big) \bigcap \Big(\Delta \times 1_{(\mathbb{P}^1)^n}\big(\mathrm{LG}(V) \times (\mathbb{P}^1)^n\big)\Big)\,. \tag{4.12}$$

The codimension of $\mathcal{Y}_{\lambda^1}\times\cdots\times\mathcal{Y}_{\lambda^n}$ in $(\mathrm{LG}(V)\times\mathbb{P}^1)^n$ is $\|\lambda^1\|+\cdots+\|\lambda^n\|$ and the dimension of $\mathrm{LG}(V)\times(\mathbb{P}^1)^n$ is $\frac{1}{2}(m^2+m)+n$. Thus the expected dimension of (4.12) is

$$\frac{1}{2}(m^2+m)+n\ -\ \|\lambda^1\|-\|\lambda^2\|-\cdots-\|\lambda^n\|\,,$$

which is the difference of $n=\dim(\mathbb{P}^1)^n$ and the number (4.11). Thus Conjecture 22 and the observed congruence modulo four would follow from a Bertini-type theorem for the families $\mathcal{Y}_{\lambda^i}$ implying that the intersection (4.12) is proper. □

The inequality (4.11) is not the final word on this congruence modulo four. When it fails, there may or may not be a congruence modulo four. We illustrate this with three examples.

The Schubert problem $\boldsymbol{\lambda}=(\yng(3,2,1),\yng(3,1,1),\yng(2,1),\yng(1),\yng(1))$ on $\mathrm{Gr}(4,8)$ has 14 solutions. We have

$$\|\yng(1)\|+\|\yng(1)\|+\|\yng(2,1)\|+\|\yng(3,1,1)\|+\|\yng(3,2,1)\|\ -\ \dim\mathrm{LG}(\mathbb{C}^8)\ =\ 1+1+2+3+4-10\ =\ 1\,,$$

so the inequality (4.11) does not hold. Table 4 shows the result of computing 200,000 instances of this problem, which took 8.7 gigaHertz-days. We observed every possible number of real solutions and thus there is no congruence modulo four for this problem.

Table 4. Schubert problem $\{(\yng(1),2)\,,\ (\yng(2,1),1)\,,\ (\yng(3,1,1),1)\,,\ (\yng(3,2,1),1)\}$ with 14 solutions.

Num. real	0	2	4	6	8	10	12	14	Total
Frequency	38008	17926	14991	4152	6938	210	6038	111737	200000

The Schubert problem $\boldsymbol{\lambda}=(\yng(3,2,1),\yng(3,2,1),\yng(2,1),\yng(1))$ on $\mathrm{Gr}(4,8)$ has 8 solutions. We have

$$\|\yng(1)\|+\|\yng(2,1)\|+\|\yng(3,2,1)\|+\|\yng(3,2,1)\|\ -\ \dim\mathrm{LG}(\mathbb{C}^8)\ =\ 1+2+4+4-10\ =\ 1\,,$$

so the inequality (4.11) does not hold. Nevertheless, we computed 400,000 instances of this problem using 265 gigaHertz-days (see Table 5), observing a congruence modulo four.

Table 5. Schubert problem $\{(\yng(1),1)\,,\ (\yng(2,1),1)\,,\ (\yng(3,1,1),2)\}$ with 8 solutions

Num. real	0	2	4	6	8	Total
Frequency	160337	0	39663	0	200000	400000

Finally, the Schubert problem $\boldsymbol{\lambda}=(\yng(3,1,1),\yng(3,1,1),\yng(2,1),TI)$ on $\mathrm{Gr}(4,8)$ has 8 solutions. We have

$$\|\yng(2,1)\|+\|\yng(2,1)\|+\|\yng(3,1,1)\|+\|\yng(3,1,1)\|+\ -\ \dim\mathrm{LG}(\mathbb{C}^8)\ =\ 2+2+3+3-10\ =\ 0\,,$$

so the inequality (4.11) does not hold. In fact $\boldsymbol{\lambda}$ gives a Schubert problem on $\mathrm{LG}(\mathbb{C}^8)$ with four solutions, so we have that $f((X_{[\boldsymbol{\mu}]})_\angle)=Z_{[\boldsymbol{\mu}]}$. Nevertheless, we computed 400,000 instances of this problem using 3.2 gigaHertz-years (see Table 6), observing a congruence modulo four.

We studied fibers of $f\colon X_{[\boldsymbol{\mu}]}\to Z_{[\boldsymbol{\mu}]}$ over points of $U_{[\boldsymbol{\mu}]}(\mathbb{R})$ for all symmetric Schubert problems on $\mathrm{Gr}(m,V)$ when $m\le 4$ whose degree $d([\boldsymbol{\mu}])$ was at most 96, a total of 44 Schubert problems in all. For each, we computed the fibers of $X_{[\boldsymbol{\mu}]}$ over several hundred thousand points in $U_{[\boldsymbol{\mu}]}(\mathbb{R})$, determining the number of real points in each fiber. These

TABLE 6. Schubert problem $\{(\yng(2,1), 2), (\yng(3,1,1), 2)\}$ with 8 solutions.

Num. real	0	2	4	6	8	Total
Frequency	147611	0	152389	0	100000	400000

data are recorded in frequency tables such as those we have given here. These are available on line [8] and are part of a larger experiment [7]. Of these, 21 satisfy the inequality (4.11) and each exhibited a congruence modulo four. (Ten satisfy the weaker condition (4.10).) Four of the remaining 23 do not satisfy the inequality (4.11), but still had the congruence modulo four, and the remaining 19 neither satisfy the inequality (4.11), nor have a congruence modulo four.

4.1. **Further Lacunae.** We close with two more symmetric Schubert problems which exhibit additional lacunae in their observed numbers of real solutions. The first is the problem $\{(\yng(3,3,3), 1), (\yng(1), 7)\}$ on $\mathrm{Gr}(4,8)$ with 20 solutions. The inequality (4.5) holds for this problem, so the possible numbers of real solutions are congruent to 20 modulo four. Table 7 displays the result of computing 400,000 instances which used 2 gigaHertz-days of

TABLE 7. Schubert problem $\{(\yng(3,3,3), 1), (\yng(1), 7)\}$ with 20 solutions.

Num. real	0	2	4	6	8	10	12	14	16	18	20	Total
Frequency	7074	0	114096	0	129829	0	0	0	0	0	119001	400000

computing. In this computation, we did not observe 12 or 16 real solutions. This is one of a family of Schubert problems on $\mathrm{Gr}(m, m+p)$ for which there are provable lower bounds and lacunae. This will be explained in [9], which describes the larger experiment [7].

Our last example is the symmetric Schubert problem $\{(\yng(1), 1), (\yng(2,1), 2), (\yng(2,2), 1), (\yng(3,1,1), 1)\}$ on $\mathrm{Gr}(4,8)$ with 16 solutions. The inequality (4.11) does not hold, so we expect every even number of solutions between 0 and 16 to occur. Table 7 displays the result of computing 200,000 instances which used 8.4 gigaHertz-days of computing. In this computation, we

TABLE 8. Schubert problem $\{(\yng(1), 1), (\yng(2,1), 2), (\yng(2,2), 1), (\yng(3,1,1), 1)\}$ with 16 solutions

Num. real	0	2	4	6	8	10	12	14	16	Total
Frequency	37069	16077	24704	10	22140	0	0	0	100000	200000

did not observe 10 or 12 or 14 real solutions. We do not know a reason for this gap in the observed number of real solutions.

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