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GEOMETRY OF JACOBI CURVES. I

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ABSTRACT. Jacobi curves are deep generalizations of the spaces of "Jacobi fields" along Riemannian geodesics. Actually, Jacobi curves are curves in the Lagrange Grassmannians. In our paper we develop differential geometry of these curves which provides basic feedback or gauge invariants for a wide class of smooth control systems and geometric structures. Two principal invariants are the *generalized Ricci curvature*, which is an invariant of the parametrized curve in the Lagrange Grassmannian endowing the curve with a natural projective structure, and a *fundamental form*, which is a fourth-order differential on the curve. The so-called rank 1 curves are studied in more detail. Jacobi curves of this class are associated with systems with scalar controls and with rank 2 vector distributions.

In the forthcoming second part of the paper we will present the comparison theorems (i.e., the estimates for the conjugate points in terms of our invariants) for rank 1 curves and introduce an important class of "flat curves."

1. INTRODUCTION

Jacobi curves were defined in [1] and [2]. Here we give a less general although more geometric construction of these curves.

Suppose that M is a smooth n-dimensional manifold and $\pi : T^*M \to M$ is the cotangent bundle to M. Let H be a codimension 1 submanifold in T^*M such that H is transversal to T_q^*M $\forall q \in M$; then $H_q = H \cap T_q^*M$ is a smooth hypersurface in T_q^*M. Let ς be the canonical Liouville form on T_q^*M, $\varsigma_\lambda = \lambda \circ \pi_*$, $\lambda \in T^*M$, and $\sigma = -d\varsigma$ be the standard symplectic structure on T^*M; then $\sigma|_H$ is a corank 1 closed 2-form. The kernels of $(\sigma|_H)_\lambda$, $\lambda \in H$ are transversal to T_q^*M, $q \in M$; these kernels form a line distribution in H and define a *characteristic* 1-*foliation* C of H. Leaves of this foliation are *characteristic curves* of $\sigma|_H$.

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Suppose that γ is a segment of a characteristic curve and O_γ is a neighborhood of γ in H such that $N = O_\gamma/(\mathcal{C}|_{O_\gamma})$ is a well-defined smooth manifold. The quotient manifold N is, in fact, a symplectic manifold endowed with a symplectic structure $\bar{\sigma}$ induced by $\sigma|_H$. Let $\phi : O_\gamma \to N$ be the canonical factorization; then $\phi(H_q \cap O_\gamma)$, $q \in M$, are Lagrangian submanifolds in N. Let $L(T_\gamma N)$ be the Lagrangian Grassmannian of the symplectic space $T_\gamma N$, i.e., $L(T_\gamma N) = \{\Lambda \subset T_\gamma N : \Lambda^\angle = \Lambda\}$, where $D^\angle = \{e \in T_\gamma N : \bar{\sigma}(e, D) = 0\}$, $\forall D \subset T_\gamma N$. *Jacobi curve of the characteristic curve* γ is the mapping

$$\lambda \mapsto \phi_*(T_\lambda H_{\pi(\lambda)}), \quad \lambda \in \gamma,$$

from γ to $L(T_\gamma N)$.

Jacobi curves are curves in the Lagrange Grassmannians. They are invariants of the hypersurface H in the cotangent bundle. In particular, any differential invariant of the curves in the Lagrange Grassmannian by the action of the linear symplectic group (i.e., any symplectic invariant) produces a well-defined function on H.

To make things clear it is not worse to give a coordinate version of the construction of the Jacobi curve. In the neighborhood O_γ, we choose coordinates $(x_0, x_1, \dots, x_{2n-2})$ such that the characteristic curves of $\sigma|_H$ are the straight lines parallel to the x_0-axis (here we do not care about the fact that H comes from the linear fiber bundle T^*M, we forget about the linear structure of the fibers). In these coordinates the sets $H_{\pi(\lambda)}$ are some $(n-1)$-dimensional submanifolds of $\mathbb{R}^{2n-1}$. For any $\lambda \in \gamma$ take projection (parallel to x_0-axis) of the spaces $T_\lambda H_{\pi(\lambda)}$ on the hyperplane $\{x_0 = c\}$ for some c. Then we obtain a curve of $(n-1)$-dimensional subspaces in the $(2n-2)$-dimensional linear space. The restriction of the form σ to $\{x_0 = c\}$ provides this space with symplectic structure and the obtained curve is a curve of Lagrangian subspaces w.r.t. this structure. This curve is exactly the Jacobi curve.

Set $W = T_\gamma N$ and note that the tangent space $T_\Lambda L(W)$ to the Lagrangian Grassmannian at the point Λ can be naturally identified with the space of quadratic forms on the linear space $\Lambda \subset W$. Namely, take a curve $\Lambda(t) \in L(W)$ with $\Lambda(0) = \Lambda$. Given some vector $l \in \Lambda$, take a curve $l(\cdot)$ in W such that $l(t) \in \Lambda(t)$ for all t and $l(0) = l$. Define the quadratic form $q_{\Lambda(\cdot)}(l) = \frac{1}{2}\bar{\sigma}\Big(\frac{d}{dt}l(0), l\Big)$. Using the fact that the spaces $\Lambda(t)$ are Lagrangian, i.e., $\Lambda(t)^\angle = \Lambda(t)$, it is easy to see that the form $q_{\Lambda(\cdot)}(l)$ depends only on $\frac{d}{dt}\Lambda(0)$. Therefore, we have the mapping from $T_\Lambda L(W)$ to the space of quadratic forms on Λ. A simple calculation of the dimension shows that this mapping is a bijection. Below we use this identification of tangent vectors to $L(W)$ with quadratic forms without a special mentioning.

Proposition 1. *Tangent vectors to the Jacobi curve J_γ at a point $J_\gamma(\lambda)$, $\lambda \in \gamma$, are equivalent (under linear substitutions of variables in the corresponding quadratic forms) to the "second fundamental form" of the hypersurface $H_{\pi(\lambda)} \subset T^*_{\pi(\lambda)}M$ at the point λ.*

Sketch of proof. In our local study we can assume without loss of generality that H is a regular level set of a smooth function h on T^*M. Then γ is a trajectory of the Hamiltonian vector field $\vec{h}$ defined by the identity $\vec{h}\rfloor\sigma = dh$. Let $t \mapsto \gamma(t)$ be a parametrization of γ defined by the Hamiltonian system $\frac{d}{dt}\gamma = \vec{h}(\gamma)$, $\gamma(0) = \lambda$. Given $l \in \phi_*(T_\lambda H_{\pi(\lambda)})$, take a vector field ℓ on H such that $\ell(\gamma(t)) \in T_{\gamma(t)}H_{\pi(\gamma(t))}$, $\phi_*\ell(\lambda) = l$. Simple calculations show that $\frac{d}{dt}\phi_*\ell(\gamma(t)) = \phi_*[\vec{h}, \ell](\gamma(t))$. Hence

$$\frac{d}{dt}J_\gamma\Big|_{t=0}(l) = \bar{\sigma}\left(\frac{d}{dt}\phi_*\ell(\gamma(t))|_{t=0}, l\right) = \sigma\big([\vec{h}, \ell](\lambda), \ell(\lambda)\big).$$

Now we rewrite the last formula in coordinates. Let $q = (q^1, \dots, q^n)$ be local coordinates in M and $p = (p_1, \dots, p_n)$ be induced coordinates in the fiber of T^*M, so that $\varsigma = \sum_{i=1}^n p_i dq^i$, $\sigma = \sum_{i=1}^n dp_i \wedge dq^i$. Then $\vec{h} = \sum_{i=1}^n \left(\frac{\partial h}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial h}{\partial q^i}\frac{\partial}{\partial p_i}\right)$, $\ell = \sum_{i=1}^n \ell^i \frac{\partial}{\partial p_i}$, and $l = (\ell^1(\lambda), \dots, \ell^n(\lambda))$. We have

$$\sigma\big([\vec{h}, \ell](\lambda), \ell(\lambda)\big) = l^* \frac{\partial^2 h}{\partial p^2} l.$$

The quadratic form $l \mapsto l^* \frac{\partial^2 h}{\partial p^2} l$ is exactly the "second fundamental form" of the hypersurface $H_{\pi(\lambda)} = h^{-1}(h(\lambda)) \cap T^*_{\pi(\lambda)}M$ in $T^*_{\pi(\lambda)}M$. $\square$

In particular, the velocity of J_γ at λ is a sign-definite quadratic form if and only if the hypersurface $H_{\pi(\lambda)}$ is strongly convex at λ.

A similar construction can be made for a submanifold of codimension 2 in T^*M. Namely, let H be a transversal to fibers of a codimension 2 submanifold in T^*M. In general, characteristic curves do not fill the whole submanifold H; they are concentrated in the characteristic variety consisting of the points, where $\sigma|_H$ is degenerate. In our local study we can always assume that H is orientable and that Ω is a volume form on M. Then $\bigwedge^{n-1}\sigma|_H = a\Omega$, where a is a smooth function on H. We set

$$C_H = \Big\{\lambda \in H : a(\lambda) = 0,\ \Big(d_\lambda a \bigwedge^{n-1}\sigma|_\lambda\Big)|_H \neq 0\Big\}.$$

Assume that $C_H \neq \emptyset$. Then C_H is a codimension 1 submanifold of H and $\sigma|_{C_H}$ is a 2-form of corank 1 on C_H. Indeed, $\forall \lambda \in C_H$, $\ker \sigma_\lambda|_H$ is a 2-dimensional subspace in $T_\lambda H$, which is transversal to $T_\lambda C_H$, and we have $\ker \sigma_\lambda|_{C_H} = \ker \sigma_\lambda|_H \cap T_\lambda C_H$.

The characteristic curves of $\sigma|_{C_H}$ form a 1-foliation $\mathcal{C}$ of C_H. Let γ be a segment of a characteristic curve and O_γ be a neighborhood of γ in H such that $N = O_\gamma/(\mathcal{C}|_{O_\gamma})$ is a well-defined smooth manifold. The quotient manifold N is a symplectic manifold endowed with a symplectic structure $\bar{\sigma}$ induced by $\sigma|_{C_H}$. Let $\phi : O_\gamma \to N$ be the canonical factorization. It is easy to verify that $\phi_*\big((T_\lambda H_{\pi(\lambda)} + \ker \sigma_\lambda|_H) \cap T_\lambda C_H\big)$ is a Lagrangian subspace of the symplectic space $T_{\phi(\lambda)}N$, $\forall \lambda \in O_\gamma$. *Jacobi curve of the characteristic curve* γ is the mapping

$$\lambda \mapsto \phi_*\big((T_\lambda H_{\pi(\lambda)} + \ker \sigma_\lambda|_H) \cap T_\lambda C_H\big), \quad \lambda \in \gamma,$$

from γ to $L(T_\gamma N)$.

We are mainly interested in submanifolds that are dual objects to smooth control systems. Here any submanifold $V \subset TM$ transversal to fibers is called a smooth control system. Let $V_q = V \cap T_q M$; the "dual" *normal variety* H^1 and *abnormal variety* H^0 are defined as follows:

$$H^1 = \bigcup_{q \in M} \big\{\lambda \in T_q^* M : \exists v \in V_q,\ \langle \lambda, v\rangle = 1,\ \langle \lambda, T_v V_q\rangle = 0\big\},$$

$$H^0 = \bigcup_{q \in M} \big\{\lambda \in T_q^* M \setminus 0 : \exists v \in V_q,\ \langle \lambda, v\rangle = \langle \lambda, T_v V_q\rangle = 0\big\}.$$

These varieties are not, in general, smooth manifolds; they may have singularities, which we do not discuss here. Anyway, one can obtain a lot of information on the original system just studying smooth parts of H^1 and H^0.

One of the varieties H^1 and H^0 can be empty. In particular, if $V_q = \partial W_q$, where W_q is a convex set and $0 \in \operatorname{int} W_q$, then $H^0 = \emptyset$. Moreover, in this case the Liouville form never vanishes on the tangent lines to the characteristic curves of $\sigma|_{H^1}$, and any characteristic curve γ has a canonical parametrization by the rule $\langle \varsigma, \dot{\gamma}\rangle = 1$. If subsets $V_q \subset T_q M$ are conical, $\alpha V_q = V_q$ $\forall \alpha > 0$, then, in contrast to the previous case, $H^1 = \emptyset$ and ς vanishes on the tangent lines to the characteristic curves of $\sigma|_{H^0}$. The characteristic curves are actually nonparametrized.

Characteristic curves of $\sigma|_{H^1}$ ($\sigma|_{H^0}$) are associated with normal (abnormal) extremals of the control system V. In [1] and [2] Jacobi curves of extremals were defined in a purely variational way in terms of the original control system and in a very general setting (including singularities), see also [6]. Introduced here Jacobi curves of characteristic curves of $\sigma|_{H^1}$

$(\sigma|_{H^0})$ coincide with Jacobi curves of the corresponding extremals in the following important cases:

(1) if H^1 has codimension 1 in T^*M. This occurs, for example, if subsets V_q are compact $\forall q \in M$;
(2) if H^0 has codimension 1 in T^*M, but $H^1 = \emptyset$. This occurs, for example, if for any q subset V_q is conical but does not contain a 2-dimensional linear space;
(3) if H^1 has codimension 2. This occurs, for example, if for any q subset V_q is an affine line in T_qM, not containing the origin;
(4) if H^0 has codimension 2. This occurs, for example , if V_q are 2-dimensional linear spaces, i.e., subsets V_q define rank 2 vector distribution on M, or if $V_q = D_q \cap \partial W_q$, where D_q is 2-dimensional linear space and W_q is a convex set such that $0 \in \operatorname{int} W_q$.

Jacobi curves associated with extremals of a given control system are not arbitrary curves of Lagrangian Grassmannian but they inherit special features of the control system. The rank of the "second fundamental form" of the submanifolds H_q^1 and H_q^0 of T_q^*M at any point is not greater than $\dim V_q$. Indeed, let $\lambda \in H_q^1$; then $\lambda \in (T_vV_q)^\perp$, $\langle \lambda, v\rangle = 1$ for some $v \in V_q$. We have $\lambda + (T_vV_q + \mathbb{R}v)^\perp \subset H_q^1$. Therefore, λ belongs to an affine subspace of dimension $n - \dim V_q - 1$, which is contained in H_q^1. For $\lambda \in H_q^0$, $\exists v \in T_qM$ such that $\lambda \in (T_vV_q)^\perp$, $\langle \lambda, v\rangle = 0$. Then linear subspace $(T_vV_q + \mathbb{R}v)^\perp$ is contained in H_q^0. It follows that the second fundamental forms of H_q^1 and H_q^0 have a rank not greater than $(\dim V_q - \operatorname{codim} H^1 + 1)$ and $(\dim V_q - \operatorname{codim} H^0 + 1)$, respectively.

In cases (1) and (2), the velocity of the Jacobi curve $\lambda \mapsto J_\gamma(\lambda)$, $\lambda \in \gamma$, associated with the extremal γ, has rank not greater than $\dim V_{\pi(\lambda)}$ (see Proposition 1). The same is true for the Jacobi curves of the extremals in cases (3) and (4), although Proposition 1 cannot be directly applied.

Dimension of V_q is the number of inputs or control parameters in the control system. Less inputs means more "nonholonomic constraints" on the system. It turns out that the rank of the velocity of any Jacobi curve generated by the system never exceeds the number of inputs.

Note that by construction these Jacobi curves are feedback invariants of the control system (i.e., they do not depend on a parametrization of the sets V_q). Hence any symplectic invariants of the Jacobi curves, associated with extremals, define a function on an appropriate submanifold of T^*M that is a feedback invariant of the control system. In this way the problem of finding feedback invariants of control systems can be reduced to a much more treatable problem of finding symplectic invariants of certain curves in the Lagrange Grassmannian.

A curve in the Lagrange Grassmannian is called regular, if its velocity at any point is a nondegenerate quadratic form. Regular curves were studied in

[1], where notions of the derivative curve and the curvature operator were introduced. Actually, the derivative curves of Jacobi curves, associated with the hypersurface H, provide a canonical connection on the cotangent bundle. If H is a spherical bundle of a Riemannian manifold, then this connection is just the Levi-Civita connection. The curvature operator of the Jacobi curve is intimately connected with the curvature tensor of the canonical connection.

In the present paper we develop general theory of curves in the Lagrangian Grassmannian. The first steps in this direction were made in [3]. It makes sense to restrict ourselves to studying so-called monotone (i.e., nondecreasing or nonincreasing) curves. The curve in Lagrangian Grassmannian is called nondecreasing (nonincreasing), if the velocity at any point of it is a nonpositive (respectively, nonnegative) quadratic form. Jacobi curve associated with the extremal of finite Morse index is automatically monotone.

This paper is organized as follows. In Sec. 2 we give the general construction of the derivative curve and introduce two principal discrete characteristics of the curves in the Lagrange Grassmannian: the rank and the weight. In particular, regular curves have the maximal rank and minimal weight. The derivative curve is defined for any curve of the finite weight. In Sec. 3 we define the curvature operator and show its role for the regular curves.

In Sec. 4 we study the *cross-ratio* of four points and an infinitesimal cross-ratio of two tangent vectors at two distinct points in the Lagrange Grassmannian. The last one leads to an intrinsic pairing $V_0, V_1 \mapsto \langle V_0 \mid V_1\rangle$, $V_i \in T_{\Lambda_i}L(W)$, $i = 0, 1$, of the tangent spaces to two distinct points Λ_0 and Λ_1 of the Grassmannian. The pairing $\langle\dot\Lambda(t) \mid \dot\Lambda(\tau)\rangle$ of the velocities of the curve $t \mapsto \Lambda(t)$ gives a symmetric function of two variables which keeps all essential information about the curve. This function is defined out of the diagonal $\{t = \tau\}$ and has a very simple singularity at the diagonal:

$$\langle\dot\Lambda(t) \mid \dot\Lambda(\tau)\rangle = -\frac{k}{(t-\tau)^2} - g_{\Lambda}(t,\tau),$$

where k is the weight of the curve and $g_{\Lambda}(t,\tau)$ is a smooth function!

The first coming invariant of the parametrized curve, the generalized Ricci curvature, is just $g_{\Lambda}(t,t)$, the value of g_{Λ} at the diagonal. For regular curves and for rank 1 curves, Ricci curvature is equal also to the trace of the defined earlier curvature operator.

In Sec. 5 we are focused on nonparametrized curves. Our investigation is based on a simple chain rule for a function g_{Λ}. Indeed, let $\varphi : \mathbb{R} \mapsto \mathbb{R}$ be a smooth monotone function. It follows directly from definition of g_{Λ} that

$$g_{\Lambda\circ\varphi}(t,\tau) = \dot\varphi(t)\dot\varphi(\tau)g_{\Lambda}(\varphi(t),\varphi(\tau)) + k\left(\frac{\dot\varphi(t)\dot\varphi(\tau)}{(\varphi(t)-\varphi(\tau))^2} - \frac{1}{(t-\tau)^2}\right).$$

In particular,

$$g_{\Lambda\circ\varphi}(t,t) = \dot\varphi(t)^2 g_\Lambda(\varphi(t),\varphi(t)) + \frac{k}{3}\mathbb{S}(\varphi),$$

where $\mathbb{S}(\varphi)$ is a Schwarzian derivative of φ. The class of local parametrizations that kill the generalized Ricci curvature defines a canonical projective structure on the curve. The principal invariant of the nonparametrized curve, the *fundamental form*, is a fourth-order differential on the curve; in the canonical projective parameter, the fundamental form has the expression $\frac{1}{2}\frac{\partial^2 g_\Lambda}{\partial\tau^2}(t,t)(dt)^4$.

In Sec. 6 we begin a systematic study of the rank 1 curves and show that a rank 1 curve has a constant weight out of a discrete set of its interval of definition. In Sec. 7 we prove that functions $\frac{\partial^{2i} g_\Lambda}{\partial\tau^{2i}}(t,t)$, $0 \le i \le m-1$, form a complete system of symplectic invariants of a rank 1 and constant weight curve $\Lambda(\cdot)$ in the Lagrange Grassmannian $L(\mathbb{R}^{2m})$.

The Lagrange Grassmannian $L(\mathbb{R}^{2m})$ is a submanifold of the manifold $G(m,2m)$ of all m-dimensional subspaces of $\mathbb{R}^{2m}$. The constructions of the derivative curve, function g_Λ, canonical projective structure, and fundamental form can be made in the same way for general curves in $G(m,2m)$.

2. Derivative curve

From now on W will be $2m$-dimensional linear space endowed with the symplectic form $\bar\sigma$. Let Λ be a Lagrangian subspace of W, i.e., $\Lambda \in L(W)$. For any $w \in \Lambda$, the linear form $\bar\sigma(\cdot,w)$ vanishes on Λ and thus defines a linear form on W/Λ. The nondegeneracy of $\bar\sigma$ implies that the relation $w \mapsto \bar\sigma(\cdot,w)$, $w \in \Lambda$, induces a canonical isomorphism $\Lambda \cong (W/\Lambda)^*$ and, by the conjugation, $\Lambda^* \cong W/\Lambda$.

We set $\Lambda^\pitchfork = \{\Gamma \in L(W) : \Gamma\cap\Lambda = 0\}$, an open everywhere dense subset of $L(W)$. Let $\mathrm{Sym}^2(\Lambda)$ be the space of self-adjoint linear mappings from Λ^* to Λ; this notation shows the fact that $\mathrm{Sym}^2(\Lambda)$ is the space of quadratic forms on Λ^* that is the symmetric square of Λ. $\Lambda^\pitchfork$ possesses a canonical structure of an affine space over the linear space $\mathrm{Sym}^2(\Lambda) = \mathrm{Sym}^2((W/\Lambda)^*)$. Indeed, for any $\Delta \in \Lambda^\pitchfork$ and coset $(w+\Lambda) \in W/\Lambda$, the intersection $\Delta \cap (w+\Lambda)$ of the linear subspace Δ and the affine subspace $w+\Lambda$ in W consists of exactly one point. To a pair $\Gamma, \Delta \in \Lambda^\pitchfork$, there corresponds a mapping $(\Gamma - \Delta) : W/\Lambda \to \Lambda$, where

$$(\Gamma - \Delta)(w+\Lambda) \stackrel{\mathrm{def}}{=} \Gamma \cap (w+\Lambda) - \Delta \cap (w+\Lambda).$$

It is easy to verify that the identification $W/\Lambda = \Lambda^*$ makes $(\Gamma - \Delta)$ a self-adjoint mapping from Λ^* to Λ. Moreover, given $\Delta \in \Lambda^\pitchfork$, the correspondence

$\Gamma \mapsto (\Gamma - \Delta)$ is a one-to-one mapping of $\Lambda^\pitchfork$ onto $\mathrm{Sym}^2(\Lambda)$ and the axioms of the affine space are obviously satisfied.

Fixing $\Delta \in \Lambda^\pitchfork$ one obtains a canonical identification $\Delta \cong W/\Lambda = \Lambda^*$. In particular, $(\Gamma - \Delta) \in \mathrm{Sym}^2(\Lambda)$ becomes the mapping from Δ to Λ. For the last linear mapping we will use the notation $\langle \Delta, \Gamma, \Lambda \rangle : \Delta \to \Lambda$. In fact, this mapping has a much more straightforward description. Namely, the relations $W = \Delta \oplus \Lambda$ and $\Gamma \cap \Lambda = 0$ imply that Γ is the graph of a linear mapping from Δ to Λ. Actually, it is the graph of the mapping $\langle \Delta, \Gamma, \Lambda \rangle$. In particular, $\ker \langle \Delta, \Gamma, \Lambda \rangle = \Delta \cap \Gamma$. If $\Delta \cap \Gamma = 0$, then $\langle \Lambda, \Gamma, \Delta \rangle = \langle \Delta, \Gamma, \Lambda \rangle^{-1}$.

Let us give coordinate representations of the introduced objects. We may assume that

$$W = \mathbb{R}^m \oplus \mathbb{R}^m = \big\{(x, y) : x, y \in \mathbb{R}^m\big\},$$

$$\bar{\sigma}((x_1, y_1), (x_2, y_2)) = \langle x_1, y_2 \rangle - \langle x_2, y_1 \rangle, \ \Lambda = \mathbb{R}^m \oplus 0, \ \Delta = 0 \oplus \mathbb{R}^m.$$

Then any $\Gamma \in \Delta^\pitchfork$ takes the form $\Gamma = \{(x, Sx) : x \in \mathbb{R}^n\}$, where S is a symmetric $m \times m$ matrix. The operator $\langle \Lambda, \Gamma, \Delta \rangle : \Lambda \to \Delta$ is represented by the matrix S, while the operator $\langle \Delta, \Gamma, \Lambda \rangle$ is represented by the matrix S^{-1}.

The coordinates in Λ induce an identification of $\mathrm{Sym}^2\Lambda$ with the space of symmetric $m \times m$ matrices. $\Lambda^\pitchfork$ is an affine subspace over $\mathrm{Sym}^2\Lambda$; we fix Δ as the origin in this affine subspace and thus obtain a coordinatization of $\Lambda^\pitchfork$ by symmetric $m \times m$ matrices. In particular, the "point" $\Gamma = \{(x, Sx) : x \in \mathbb{R}^n\}$ in $\Lambda^\pitchfork$ is represented by the matrix S^{-1}.

A subspace $\Gamma_0 = \{(x, S_0 x) : x \in \mathbb{R}^n\}$ is transversal to Γ if and only if $\det(S - S_0) \neq 0$. Let us choose coordinates $\{x\}$ in Γ_0 and fix Δ as the origin in the affine space $\Gamma_0^\pitchfork$. In the induced coordinatization of $\Gamma_0^\pitchfork$ the "point" Γ is represented by the matrix $(S - S_0)^{-1}$.

Let $t \mapsto \Lambda(t)$ be a smooth curve in $L(W)$ defined on some interval $I \subset \mathbb{R}$. We say that the curve $\Lambda(\cdot)$ is *ample at* τ if $\exists s > 0$ such that for any representative $\Lambda_\tau^s(\cdot)$ of the s-jet of $\Lambda(\cdot)$ at τ, $\exists t$ such that $\Lambda_\tau^s(t) \cap \Lambda(\tau) = 0$. The curve $\Lambda(\cdot)$ is called *ample* if it is ample at any point.

We have given an intrinsic definition of an ample curve. In coordinates it takes the following form: the curve $t \mapsto \{(x, S_t x) : x \in \mathbb{R}^n\}$ is ample at τ if and only if the function $t \mapsto \det(S_t - S_\tau)$ has a root of *finite order* at τ.

The following lemma shows that analytic monotone curve (monotone means that the curve has nonnegative or nonpositive velocities at all points) can be actually reduced to the ample curve by an appropriate factorization.

Lemma 2.1. *Let $\Lambda(t)$ be an analytic monotone curve in $L(W)$. Then for any parameter τ there exists a subspace K of $\Lambda(\tau)$ such that for all t sufficiently close to τ the following relation holds*:

$$K = \Lambda(t) \cap \Lambda(\tau). \tag{2.1}$$

In addition, if $\Lambda(t)$ is not a constant curve, then the curve $t \mapsto \Lambda(t)/K$ is a well-defined ample curve in the Lagrange Grassmannian $L(K^{\angle}/K)$.

Proof. Without loss of generality suppose that the curve $\Lambda(t)$ is nondecreasing (i.e., has a nonnegative definite velocity at any point). Denote $K_t = \Lambda(t) \cap \Lambda(\tau)$. Let $t \mapsto \{(x, S_t x) : x \in \mathbb{R}^n\}$ be a coordinate representation of the germ of $\Lambda(t)$ at τ. Then $K_t = \operatorname{Ker}(S_t - S_\tau)$. By assumption, $v \to \left\langle \frac{d}{dt} S_t v, v \right\rangle$ is a nonnegative definite quadratic form on $\Lambda(\tau)$. This implies that $K_{t_2} \subset K_{t_1}$ for $t < t_1 < t_2$. Therefore, for $t > \tau$ sufficiently close to τ the subspace K_t does not depend on t and will be denoted by K. By analyticity, the subspaces $K \subset \Lambda(t)$ for any t and the curve $t \mapsto \Lambda(t)/K$ is a well-defined ample curve in the Lagrange Grassmannian $L(K^{\angle}/K)$. $\square$

Assume that $\Lambda(\cdot)$ is ample at τ. Then $\Lambda(t) \in \Lambda(\tau)^{\pitchfork}$ for all t from a punctured neighborhood of τ. We obtain the curve $t \mapsto \Lambda(t) \in \Lambda(\tau)^{\pitchfork}$ in the affine space $\Lambda(\tau)^{\pitchfork}$ with the pole at τ. We denote by $\Lambda_\tau(t)$ the identity imbedding of $\Lambda(t)$ in the affine space $\Lambda(\tau)^{\pitchfork}$. The subscript τ is not superfluous, since the affine structure depends on $\Lambda(\tau)$ and, therefore, on τ. Fixing an "origin" in $\Lambda(\tau)^{\pitchfork}$, we make $\Lambda_\tau(t)$ a vector function with values in $\operatorname{Sym}^2(\Lambda)$ and with the pole at $t = \tau$. Such a vector function admits an expansion into the Laurent series at τ. Obviously, only the free term in the Laurent expansion depends on the choice of the "origin" chosen for identifying the affine space with the linear one. More precisely, the addition of a vector to the "origin" results in the addition of the same vector to the free term in the Laurent expansion. In other words, for the Laurent expansion of a curve in an affine space, the free term of the expansion is a point of this affine space while all other terms are elements of the corresponding linear space. In particular,

$$\Lambda_\tau(t) \approx \Lambda^0(\tau) + \sum_{\substack{i=-l \\ i \neq 0}}^{\infty} Q_i(\tau)(t-\tau)^i, \tag{2.2}$$

where $\Lambda^0(\tau) \in \Lambda(\tau)^{\pitchfork}$, $Q_i(\tau) \in \operatorname{Sym}^2 \Lambda(\tau)$.

Assume that the curve $\Lambda(\cdot)$ is ample. Then $\Lambda^0(\tau) \in \Lambda(\tau)^{\pitchfork}$ is defined for all τ. The curve $\tau \mapsto \Lambda^0(\tau)$ is called the *derivative curve* of $\Lambda(\cdot)$.

Another characterization of $\Lambda^0(\tau)$ can be given in terms of the curves $t \mapsto \langle \Delta, \Lambda(t), \Lambda(\tau) \rangle$ in the linear space $\operatorname{Hom}(\Delta, \Lambda(\tau))$, $\Delta \in \Lambda(\tau)^{\pitchfork}$. These curves have poles at τ. The Laurent expansion at $t = \tau$ of the vector function $t \mapsto \langle \Delta, \Lambda(t), \Lambda(\tau) \rangle$ has zero free term if and only if $\Delta = \Lambda^0(\tau)$.

The coordinate version of series (2.2) is the Laurent expansion of the matrix-valued function $t \mapsto (S_t - S_\tau)^{-1}$ at $t = \tau$, where $\Lambda(t) = \{(x, S_t x) : x \in \mathbb{R}^n\}$.

Suppose that

$$(S_t - S_\tau)^{-1} \approx \sum_{i=-l}^{\infty} A_i(\tau)(t-\tau)^i. \tag{2.3}$$

Differentiating both sides of (2.3) w.r.t. τ and comparing coefficients of the corresponding expansions, one can obtain the following recursive type formula for the coefficients $A_i(\tau)$:

$$\frac{d}{d\tau} A_i(\tau) = (i+1)A_{i+1}(\tau) + \sum_{j=-l}^{i+l} A_j(\tau)\dot S_\tau A_{i-j}(\tau) \tag{2.4}$$

that will be used in the sequel.

For a monotone ample curve $\Lambda : I \subset \mathbb{R} \mapsto L(W)$ we introduce the following two notions.

Definition 1. The rank of the velocity $\dot\Lambda(\tau)$ is said to be the *rank* of the curve $\Lambda(\cdot)$ at τ. The order of the zero of the function $t \mapsto \det(S_t - S_\tau)$ at τ, where S_t is a coordinate representation of $\Lambda(\cdot)$, is said to be the *weight* of $\Lambda(\cdot)$ at τ.

It is easy to see that the rank and the weight of $\Lambda(\tau)$ are integer valued upper semicontinuous functions of τ. In particular, they are locally constant on the open dense subset of the interval of definition I. In the sequel we will be mostly concentrated on the monotone ample curves of the constant rank and weight.

3. Curvature operator and regular curves

Using derivative curve, one can construct an operator invariant of the curve $\Lambda(t)$ at any its point. Namely, take velocities $\dot\Lambda(t)$ and $\dot\Lambda^0(t)$ of $\Lambda(t)$ and its derivative curve $\Lambda^0(t)$. Note that $\dot\Lambda(t)$ is a linear operator from $\Lambda(t)$ to $\Lambda(t)^*$ and $\dot\Lambda^0(t)$ is a linear operator from $\Lambda^0(t)$ to $\Lambda^0(t)^*$. Since the form σ defines the canonical isomorphism between $\Lambda^0(t)$ and $\Lambda(t)^*$, we can define the following operator $R(t) : \Lambda(t) \to \Lambda(t)$:

$$R(t) = -\dot\Lambda^0(t) \circ \dot\Lambda(t). \tag{3.1}$$

This operator is said to be the *curvature operator* of Λ at t.

Remark 1. In the case of the Riemannian geometry, the operator $R(t)$ is similar to the so-called Ricci operator $v \to R^\nabla(\dot\gamma(t), v)\dot\gamma(t)$, which appears in the classical Jacobi equation $\nabla_{\dot\gamma(t)}\nabla_{\dot\gamma(t)}V + R^\nabla(\dot\gamma(t), V)\dot\gamma(t) = 0$ for Jacobi vector fields V along the geodesic $\gamma(t)$ (here R^∇ is curvature tensor of the Levi-Civita connection ∇), see [1]. This is the reason for the sign "$-$" in (3.1).

The curvature operator plays an important role for so-called regular curves. The curve $\Lambda(t)$ in the Lagrangian Grassmannian is called *regular*, if the quadratic form $\dot\Lambda(t)$ is nondegenerate for all t. If the curve $\Lambda(\cdot)$ is regular and has a coordinate representation $\Lambda(t) = \big\{(x, S_t x) : x \in \mathbb{R}^n\big\}$, then the function $t \mapsto (S_t - S_\tau)^{-1}$ has a simple pole at τ. Indeed,

$$(S_t - S_\tau)^{-1} = \Big(\dot S_\tau (t-\tau) + O((t-\tau)^2)\Big)^{-1} = \frac{\dot S_\tau^{-1}}{t-\tau}\Big(I + O(t-\tau)\Big)^{-1} = \\ = \frac{\dot S_\tau^{-1}}{t-\tau} + O(1). \tag{3.2}$$

Therefore, in the notation of (2.3), for the regular curve we have $l = 1$, $A_{-1} = \dot S_\tau^{-1}$, and relation (2.4) can be transformed into the following recursive formula:

$$A_{i+1}(\tau) = \frac{1}{i+3}\left(\frac{d}{d\tau}A_i(\tau) - \sum_{j=0}^{i} A_j(\tau)\dot S_\tau A_{i-j}(\tau)\right). \tag{3.3}$$

In particular,

$$A_0(\tau) = \frac{1}{2}\frac{d}{d\tau}A_{-1}(\tau) = -\frac{1}{2}\dot S_\tau^{-1}\ddot S_\tau \dot S_\tau^{-1} \tag{3.4}$$

and, by a direct calculation,

$$A_1(\tau) = \frac{1}{3}\left(\frac{d}{d\tau}A_0(\tau) - A_0(\tau)\dot S_\tau A_0(\tau)\right) = \\ = \left(\frac{1}{4}\left(\dot S_\tau^{-1}\ddot S_\tau\right)^2 - \frac{1}{6}\dot S_\tau^{-1} S_\tau^{(3)}\right)\dot S_\tau^{-1}. \tag{3.5}$$

For a given τ one can choose a coordinate representation S_t of the curve $\Lambda(t)$ such that $A_0(\tau) = 0$. Namely, let S_t be the matrix of the linear mapping $\langle \Lambda(\tau), \Lambda(t), \Lambda^0(\tau)\rangle$. In this coordinate representation, the derivative $\dot A_0(\tau)$ is a matrix corresponding to the velocity $\dot\Lambda^0(\tau)$ of the derivative curve. Also, from (3.5) it follows that $\dot A_0(\tau) = 3A_1(\tau)$. This together with (3.1) implies that the matrix $\mathcal{R}(\tau)$ corresponding in the chosen basis of $\Lambda(\tau)$ to the curvature operator $R(\tau)$ has the form

$$\mathcal{R}(\tau) = -3A_1(\tau)(A_{-1}(\tau))^{-1} = \frac{1}{2}\dot S_\tau^{-1} S_\tau^{(3)} - \frac{3}{4}\left(\dot S_\tau^{-1}\ddot S_\tau\right)^2 = \\ = \frac{d}{d\tau}\left((2\dot S_\tau)^{-1}\ddot S_\tau\right) - \left((2\dot S_\tau)^{-1}\ddot S_\tau\right)^2. \tag{3.6}$$

Since $Q_1(\tau) \circ (Q_{-1}(\tau))^{-1} : \Lambda(\tau) \mapsto \Lambda(\tau)$ is a well-defined operator, we can write the first equality of (3.6) in the operator form:

$$R(\tau) = Q_1(\tau) \circ (Q_{-1}(\tau))^{-1}. \tag{3.7}$$

This actually implies that relation (3.6) also holds for any coordinate representation S_t of the curve $\Lambda(t)$ (even without assumption that $A_0(\tau)=0$).

Note that the right-hand side of (3.6) is a matrix analog of the so-called Schwarz derivative or Schwarzian. Let us recall that the differential operator

$$\mathbb{S}:\varphi\mapsto \frac{1}{2}\frac{\varphi^{(3)}}{\varphi'}-\frac{3}{4}\Big(\frac{\varphi''}{\varphi'}\Big)^2=\frac{d}{dt}\Big(\frac{\varphi''}{2\,\varphi'}\Big)-\Big(\frac{\varphi''}{2\,\varphi'}\Big)^2 \tag{3.8}$$

acting on scalar functions φ is called *Schwarzian*. The operator $\mathbb{S}$ is characterized by the following remarkable property: a general solution of the equation $\mathbb{S}\varphi=\rho$ w.r.t. φ is a Möbius transformation (with constant coefficients) of some particular solution of this equation. The matrix analog of this operator has a similar property, concerning "matrix Möbius transformations" of the type $S\mapsto(C+DS)(A+BS)^{-1}$. In particular, if $R(t)\equiv 0$, then the coordinate representation S_t of our curve has the form

$$S_t=(C+Dt)(A+Bt)^{-1},$$

where

$$\begin{pmatrix} A & B\\ C & D\end{pmatrix}\in \mathrm{Sp}(2m).$$

For further information about the regular curves we refer to [1].

4. Expansion of the cross-ratio and Ricci curvature

For a nonregular curve $\Lambda(t)=\big\{(x,S_tx):x\in\mathbb{R}^n\big\}$, the function $t\mapsto(S_t-S_\tau)^{-1}$ has a pole of order greater than 1 at τ and it is much more difficult to compute its Laurent expansion. For example, in the nonregular case there is no direct recursive formula like (3.3). In this section we show how to construct numerical invariants for curves of constant weight using the notion of the cross-ratio of four "points" in the Lagrange Grassmannian.

Let Λ_0, Λ_1, Λ_2, and Λ_3 be Lagrangian subspaces of W such that $\Lambda_0\cap\Lambda_3=\Lambda_1\cap\Lambda_2=0$. Also suppose for simplicity that $\Lambda_0\cap\Lambda_2=0$. The linear mappings $\langle\Lambda_0,\Lambda_1,\Lambda_2\rangle:\Lambda_0\to\Lambda_2$ and $\langle\Lambda_2,\Lambda_3,\Lambda_0\rangle:\Lambda_2\to\Lambda_0$ are well defined. The cross-ratio $\Big[\Lambda_0,\Lambda_1,\Lambda_2,\Lambda_3\Big]$ of four "points" Λ_0, Λ_1, Λ_2, and Λ_3 in the Lagrangian Grassmannian is, by definition, the following linear operator in Λ_0:

$$\Big[\Lambda_0,\Lambda_1,\Lambda_2,\Lambda_3\Big]=\langle\Lambda_2,\Lambda_3,\Lambda_0\rangle\langle\Lambda_0,\Lambda_1,\Lambda_2\rangle. \tag{4.1}$$

This notion is a "matrix" analog of the classical cross-ratio of four points in the projective line. Indeed, let $\Lambda_i=\{(x,S_ix):x\in\mathbb{R}^n\}$, then, in coordinates $\{x\}$, the cross-ratio takes the form:

$$[\Lambda_0,\Lambda_1,\Lambda_2,\Lambda_3]=(S_0-S_3)^{-1}(S_3-S_2)(S_2-S_1)^{-1}(S_1-S_0). \tag{4.2}$$

By construction, all coefficients of the characteristic polynomial of $[\Lambda_0\,,\,\Lambda_1,\ \Lambda_2,\ \Lambda_3]$ are invariants of four subspaces Λ_0, Λ_1, Λ_2, and Λ_3.

The assumption that $\Lambda_0 \cap \Lambda_2 = 0$ is satisfied in our further considerations but the cross-ratio can also be defined without this assumption. Indeed, the matrix in the right-hand side of (4.2) is also well defined in the case $\Lambda_0 \cap \Lambda_2 \neq 0$ and this matrix is transformed into a similar matrix under any change of coordinates. Thus we obtain the class of similar matrices that is a symplectic invariant of four subspaces Λ_0, Λ_1, Λ_2, and Λ_3. This class can be taken as a definition of the cross-ratio $[\Lambda_0, \Lambda_1, \Lambda_2, \Lambda_3]$ (see [7] for details).

Given two tangent vectors $V_0 \in T_{\Lambda_0}L(W)$ and $V_1 \in T_{\Lambda_1}L(W)$, where Λ_0 and Λ_1 are transversal Lagrangian subspaces, one can define an infinitesimal analog of the cross-ratio. V_0 is the self-adjoint linear mapping from Λ_0 to Λ_0^*. The form σ identifies canonically Λ_0^* with Λ_1. Under this identification V_0 can be considered as a linear mapping from Λ_0 to Λ_1. In the same way, identifying Λ_1^* with Λ_0, we look at V_1 as at an operator from Λ_1 to Λ_0. Therefore, the following operator $V_1 \odot V_0 : \Lambda_0 \to \Lambda_0$ can be defined:

$$V_1 \odot V_0 \stackrel{\text{def}}{=} V_1 \circ V_0. \tag{4.3}$$

This operator is said to be an *infinitesimal cross-ratio* of a pair $(V_0, V_1) \in T_{\Lambda_0}L(W) \times T_{\Lambda_1}L(W)$. The infinitesimal cross-ratio is a symplectic invariant of the tangent vectors V_0 and V_1.

One can define the following bilinear form $\langle \cdot \mid \cdot \rangle_{\Lambda_0,\Lambda_1}$ on $T_{\Lambda_0}L(W) \times T_{\Lambda_1}L(W)$:

$$\langle V_0 \mid V_1 \rangle_{\Lambda_0,\Lambda_1} \stackrel{\text{def}}{=} \operatorname{tr}(V_0 \odot V_1). \tag{4.4}$$

This bilinear form is said to be an *inner pairing* of the tangent spaces $T_{\Lambda_0}L(W)$ and $T_{\Lambda_1}L(W)$.

If $\Lambda_i = \big\{(x, S_i x) : x \in \mathbb{R}^n\big\}$ and P_i are symmetric matrices corresponding to V_i, $i = 0, 1$, then

$$V_1 \odot V_0 = (S_0 - S_1)^{-1} P_1 (S_1 - S_0)^{-1} P_0. \tag{4.5}$$

We note first that if the curve $\Lambda(t)$ is regular, then for any t_0 it is easy to expand the following operator function

$$(t_1, t_2, t_3) \mapsto \frac{[\Lambda(t_0), \Lambda(t_1), \Lambda(t_2), \Lambda(t_3)]}{[t_0, t_1, t_2, t_3]} \tag{4.6}$$

into the Taylor expansion at the diagonal point (t_0, t_0, t_0), where

$$\Big[t_0, t_1, t_2, t_3\Big] = \frac{(t_1 - t_0)(t_3 - t_2)}{(t_2 - t_1)(t_0 - t_3)}$$

is the usual cross-ratio of four numbers t_0, t_1, t_2, and t_3. Namely, the expansion

$$\frac{[\Lambda(t_0),\Lambda(t_1),\Lambda(t_2),\Lambda(t_3)]}{[t_0,t_1,t_2,t_3]} =$$
$$= I + \frac{1}{3}R(t_0)(t_2-t_0)(t_3-t_1) + O\biggl(\Bigl(\sum_{i=1}^{3}(t_i-t_0)^2\Bigr)^{3/2}\biggr) \tag{4.7}$$

is valid, where, as before, $R(t)$ is the curvature operator. Relation (4.7) shows that the curvature operator is the first nontrivial coefficient of the Taylor expansion of the cross-ratio.

Unfortunately, for the nonregular curves there are no simple expansions of the operator function (4.6) or any other operator functions, involving the cross-ratio itself. Instead of this one can try to expand the coefficients of the characteristic polynomial of the cross-ratio. Now we are going to show how to use this idea in the construction of invariants of the curve $\Lambda(t)$ of the constant weight k in $L(W)$.

By the above, the function $(t_0,t_1,t_2,t_3) \to \det\,[\Lambda(t_0),\Lambda(t_1),\Lambda(t_2),\Lambda(t_3)]$ is one of the symplectic invariants of the curve $\Lambda(t)$. Using this fact, let us try to find symplectic invariants of $\Lambda(t)$ that are functions of t. First, we introduce the following function:

$$\mathcal{G}(t_0,t_1,t_2,t_3) = \ln\left(\frac{\det\bigl[\Lambda(t_0),\Lambda(t_1),\Lambda(t_2),\Lambda(t_3)\bigr]}{\bigl[t_0,t_1,t_2,t_3\bigr]^k}\right). \tag{4.8}$$

The function $\mathcal{G}(t_0,t_1,t_2,t_3)$ is also a symplectic invariant of $\Lambda(t)$ and, in addition, it can be defined as a smooth function in a neighborhood of any diagonal point (t,t,t,t). Indeed, by the definition of the weight

$$\det(S_{t_0}-S_{t_1}) = (t_0-t_1)^k X(t_0,t_1), \tag{4.9}$$

where

$$X(t,t) \neq 0 \tag{4.10}$$

for any t. The function $X(t_0,t_1)$ is symmetric, since by changing the order in (4.9) we obtain that X can be symmetric or antisymmetric, but the last case is impossible by (4.10).

Let us define another symmetric function

$$f(t_0,t_1) = \ln X(t_0,t_1). \tag{4.11}$$

The function $f(t_0,t_1)$ is smooth in a neighborhood of any diagonal point (t,t) and by (4.2) and (4.8),

$$\mathcal{G}(t_0,t_1,t_2,t_3) = f(t_1,t_0) - f(t_2,t_1) + f(t_3,t_2) - f(t_0,t_3). \tag{4.12}$$

Hence $\mathcal{G}(t_0, t_1, t_2, t_3)$ can be defined as a smooth function in a neighborhood of any diagonal point (t, t, t, t). Using this fact, one can construct the following functions of two variables that are symplectic invariants of the curve $\Lambda(t)$:

$$h(t_0, t_1) = \mathcal{G}(t_0, t_1, t_1, t_0) = 2f(t_0, t_1) - f(t_0, t_0) - f(t_1, t_1), \quad (4.13)$$

$$g(t_0, t_1) = \frac{1}{2}\frac{\partial^2}{\partial t_0 \partial t_1} h(t_0, t_1) = \frac{\partial^2}{\partial t_0 \partial t_1} f(t_0, t_1). \quad (4.14)$$

On the contrary, the function $f(t_0, t_1)$ depends on the choice of the coordinate representation S_t.

It follows from (4.13) that $h(t_0, t_0) \equiv 0$ and $\frac{\partial}{\partial t_0} h(t_0, t_0) \equiv 0$. Therefore, the function $h(t_0, t_1)$ can be recovered from $g(t_0, t_1)$. Moreover, the function $\mathcal{G}(t_0, t_1, t_2, t_3)$ can be easily recovered from $h(t_0, t_1)$ (and, therefore, from $g(t_0, t_1)$). Namely, by (4.12) and (4.14)

$$\mathcal{G}(t_0, t_1, t_2, t_3) = \frac{1}{2}\big(h(t_1, t_0) - h(t_2, t_1) + h(t_3, t_2) - h(t_0, t_3)\big). \quad (4.15)$$

Therefore, g or h keep all information about $\mathcal{G}$ and thus about

$$\det\Big[\Lambda(t_0), \Lambda(t_1), \Lambda(t_2), \Lambda(t_3)\Big].$$

The function $g(t_0, t_1)$ can be expanded into a formal Taylor series at the point (t, t) in the following way:

$$g(t_0, t_1) \approx \sum_{i,j=0}^{\infty} \beta_{i,j}(t)(t_0 - t)^i (t_1 - t)^j \quad (4.16)$$

with

$$\beta_{i,j}(t) = \beta_{j,i}(t). \quad (4.17)$$

Since the function g is a symplectic invariant of the curve $\Lambda(t)$, all coefficients $\beta_{i,j}(t)$, $i, j \geq 0$, are also symplectic invariants.

The following natural questions arise: does the function $g(t_0, t_1)$ determine the curve $\Lambda(t)$ with a prescribed rank and weight uniquely, up to a symplectic transformation, and what set of the coefficients $\beta_{i,j}(t)$ determines the function $g(t_0, t_1)$? We will give the positive answers on both of these questions in Sec. 7 for the curve of rank 1 (see Theorems 1 and 2).

Meanwhile, let us prove the following simple relation between coefficients $\beta_{i,j}(t)$:

$$\beta'_{i,j}(t) = (i + 1)\beta_{i+1,j} + (j + 1)\beta_{i,j+1}. \quad (4.18)$$

Indeed, from (4.16) it follows that

$$\beta_{i,j}(t) = \frac{1}{i!\,j!}\frac{\partial^{i+j} g}{\partial t_0^i \partial t_1^j}(t,t).$$

Therefore,

$$\begin{aligned}\beta'_{i,j}(t) &= \frac{1}{i!\,j!}\Big(\frac{\partial^{i+j+1}}{\partial t_0^{i+1}\partial t_1^j} g(t,t) + \frac{\partial^{i+j+1}}{\partial t_0^{i}\partial t_1^{j+1}} g(t,t)\Big) = \\ &= \frac{1}{i!\,j!}\Big((i+1)!\,j!\,\beta_{i+1,j}(t) + i!\,(j+1)!\,\beta_{i,j+1}(t)\Big),\end{aligned}$$

which implies (4.18).

As a consequence of relation (4.18) we obtain the following lemma.

Lemma 4.1. *The coefficients* $\beta_{0,2k}(t)$, $k \geq 0$, *determine uniquely the formal expansion* (4.16).

Proof. For a given $n \geq 0$ let us consider all equations of type (4.18) with $i + j = n$ and $i \leq j$. We consider two cases:

(1) if n is even, then we have $\frac{n}{2} + 1$ independent equations with respect to $\frac{n}{2} + 1$ variables $\beta_{i,j}(t)$, $i + j = n + 1$, $0 \leq i < \frac{n}{2}$. This fact together with symmetric relation (4.17) implies that all $\beta_{i,j}(t)$ with $i + j = n + 1$ can be expressed in terms of derivatives of $\beta_{i,j}$ with $i + j = n$;

(2) if n is odd, then we have $\frac{n+1}{2}$ independent equations with respect to $\frac{n+1}{2} + 1$ variables $\beta_{i,j}$, $i + j = n + 1$, $0 \leq i < \frac{n+1}{2}$. Starting from $i = 0$, one can express step by step all $\beta_{i,j}$, $i + j = n + 1$, $1 \leq i < \frac{n+1}{2}$ in terms of $\beta_{0,n+1}$ and derivatives of $\beta_{i,j}$ with $i + j = n$. Then by symmetric relation (4.17) we have that all coefficients $\beta_{i,j}(t)$ with $i + j = n + 1$ can be expressed in terms of $\beta_{0,n+1}$ and derivatives of $\beta_{i,j}$ with $i + j = n$.

Therefore, starting from $n = 0$ and applying step by step the arguments of (1) and (2), one can express all $\beta_{i,j}(t)$ in terms of $\beta_{0,2k}(t)$, $k \geq 0$, and their derivatives. $\square$

It turns out that there is a simple connection between function g, the inner pairing defined by (4.4), and the coefficients Q_i of the Laurent expansion (2.2).

Lemma 4.2. *The following relations hold*:

$$\langle\, \dot\Lambda(t) \mid \dot\Lambda(\tau)\,\rangle_{\Lambda(t),\Lambda(\tau)} = -\frac{k}{(t-\tau)^2} - g(t,\tau), \tag{4.19}$$

$$\operatorname{tr}\bigl(Q_i(t)\dot\Lambda(t)\bigr) = 0, \quad i < -1, \tag{4.20}$$

$$\operatorname{tr}\bigl(Q_{-1}(t)\dot\Lambda(t)\bigr) = k, \tag{4.21}$$

$$\operatorname{tr}\bigl(Q_i(t)\dot\Lambda(t)\bigr) = -\frac{1}{i}\beta_{0,i-1}(t), \quad i \in \mathbb{N}. \tag{4.22}$$

Proof. Let $\Lambda_\tau(t)$ be the identity imbedding of $\Lambda(t)$ in the affine space $\Lambda(\tau)^{\pitchfork}$ (see Sec. 2). Then the inner pairing $\langle\, \dot\Lambda(t) \mid \dot\Lambda(\tau)\,\rangle_{\Lambda(t),\Lambda(\tau)}$ can be expressed as follows:

$$\langle\, \dot\Lambda(t) \mid \dot\Lambda(\tau)\,\rangle_{\Lambda(t),\Lambda(\tau)} = \operatorname{tr}\left(\frac{\partial}{\partial t}\Lambda_\tau(t) \circ \dot\Lambda(\tau)\right). \tag{4.23}$$

In the coordinates, the previous relation can be written as follows:

$$\langle\, \dot\Lambda(t) \mid \dot\Lambda(\tau)\,\rangle_{\Lambda(t),\Lambda(\tau)} = \operatorname{tr}\left(\frac{\partial}{\partial t}\bigl((S_t - S_\tau)^{-1}\bigr)\dot S_\tau\right). \tag{4.24}$$

Let us prove (4.20). By definition,

$$\ln(\det(S_t - S_\tau)) = k\ln(t-\tau) + f(t,\tau).$$

Differentiating this equality w.r.t. τ and using the fact that

$$\frac{d}{d\tau}\bigl(\ln(\det Y(\tau))\bigr) = \operatorname{tr}\bigl((Y(\tau))^{-1}\dot Y(\tau)\bigr)$$

for some matrix curve $Y(\tau)$, we obtain:

$$-\operatorname{tr}\bigl((S_t - S_\tau)^{-1}\dot S_\tau\bigr) = -\frac{k}{t-\tau} + \frac{\partial}{\partial\tau} f(t,\tau).$$

Differentiating this equality w.r.t. t and using (4.14), we obtain

$$-\operatorname{tr}\left(\frac{\partial}{\partial t}\left((S_t - S_\tau)^{-1}\right)\dot S_\tau\right) = \frac{k}{(t-\tau)^2} + \frac{\partial^2}{\partial t\partial\tau} f(t,\tau) = \\ = \frac{k}{(t-\tau)^2} + g(t,\tau).$$

This together with (4.24) implies (4.20).

In order to prove (4.20)–(4.22), we expand both sides of (4.20) into the corresponding formal series. On one hand, by (2.2) we have

$$\operatorname{tr}\left(\frac{\partial}{\partial t}\Lambda_\tau(t) \circ \dot\Lambda(\tau)\right) \approx \sum_{i=-l-1}^{\infty} (i+1)\operatorname{tr}\bigl(Q_{i+1}(\tau)\dot\Lambda(\tau)\bigr)(t-\tau)^i. \tag{4.25}$$

On the other hand, by (4.16)

$$g(t,\tau) \approx \sum_0^\infty \beta_{0,i}(\tau)(t-\tau)^i. \tag{4.26}$$

Comparing coefficients of (4.25) and (4.26), we obtain (4.20)–(4.22). □

For the regular curve, using (3.6) and applying formula (4.22) to the first appearing in (4.16) coefficient $\beta_{0,0}(t)$, we obtain

$$\beta_{0,0}(t) = \frac{1}{3}\operatorname{tr} R(t) = \frac{1}{3}\operatorname{tr}\mathbb{S}(S_t), \tag{4.27}$$

where $\mathbb{S}$ denotes Schwarz operator. The last relation and Remark 1 show that $\beta_{0,0}$ generalizes the Ricci curvature in the Riemannian geometry. It justifies the following definition for the general curve of constant rank and weight.

Definition 2. The first appearing in (4.16) coefficient $\beta_{0,0}(t)$ is called the Ricci curvature of $\Lambda(t)$.

In the sequel, the Ricci curvature will be denoted by $\rho(t)$.

At the end of this section we compute the expansion of $g(t_0,t_1)$ in the case $\dim W = 2$. In this case $L(W)$ is, in fact, the real projective line $\mathbb{RP}^1$ and coordinate representation S_t of the curve is a scalar function. Therefore, relation (4.22) can be rewritten in the form

$$\beta_{0,i} = -(i+1)A_{i+1}(t)\dot S_t,$$

where A_i are as in (2.3). In particular, from (4.27) it follows that

$$\rho(t) = \frac{1}{3}\mathbb{S}(S_t), \tag{4.28}$$

i.e., in the scalar case the Ricci curvature of the curve $\Lambda(t)$ is the Schwarzian of its coordinate representation.

Let us denote

$$B_i(\tau) = -\frac{1}{i}\beta_{0.i-1} = A(\tau)\dot S_\tau. \tag{4.29}$$

Multiplying both sides of (3.3) by $\dot S_\tau$ and using the commutativity of multiplication in the scalar case, one can easily obtain the following recursive formula for$B_i(\tau)$:

$$B_{i+1}(\tau) = \frac{1}{i+3}\left(\frac{d}{d\tau}B_i(\tau) - \sum_{j=1}^{i-1} B_j(\tau)B_{i-j}(\tau)\right), \quad i \in \mathbb{N}. \tag{4.30}$$

As a consequence of Lemma 4.1 and formulas (4.28)–(4.30), one can obtain the following proposition.

Proposition 2. *In the scalar case (i.e.,* $\dim W = 2$*) all coefficients* $\beta_{i,j}(t)$ *can be expressed in terms of the Ricci curvature (that is Schwarzian of any coordinate representation of the curve* $\Lambda(t)$*) and its derivatives. The function* $g(t,\tau)$ *is identically equal to zero iff coordinate representations of the curve* $\Lambda(t)$ *are Möbius transformations.*

5. Fundamental form of the nonparametrized curve

The Jacobi curve constructed in Introduction is actually a nonparametrized curve, i.e., a one-dimensional submanifold in the Lagrange Grassmannian. Therefore, it is natural to find symplectic invariants of nonparametrized curves in $L(W)$. Especially it is important for Jacobi curves of abnormal extremals which (in contrast to the normal extremals) a priori have no special parametrizations.

First of all, we want to show how, using the Ricci curvature, one can define a canonical projective structure on the nonparametrized curve $\Lambda(\cdot)$. For this let us find how the Ricci curvature is transformed by a reparametrization of the curve $\Lambda(t)$.

Let $\tau = \varphi(t)$ be a reparametrization and let $\bar{\Lambda}(\tau) = \Lambda(\varphi^{-1}(\tau))$. For some coordinate representation S_t of $\Lambda(t)$ let $\bar{S}_\tau = S_{\varphi^{-1}(\tau)}$ be the coordinate representation of $\bar{\Lambda}(\tau)$. Denote by $\bar{f}$ the function playing for $\bar{S}_\tau$ the same role as the function f defined by (4.11) plays for S_t. Then from (4.11) it follows that

$$\bar{f}(\tau_0,\tau_1) = f(t_0,t_1) - k\,\ln\left(\frac{\varphi(t_0)-\varphi(t_1)}{t_0-t_1}\right), \tag{5.1}$$

where $\tau_i = \varphi(t_i)$, $i = 0,1$.

Now we denote by $\bar{g}$ and $\bar{\beta}_{i,j}$ functions playing for $\bar{\Lambda}(\tau)$ the same role as the functions g and $\beta_{i,j}$ defined by (4.14) and (4.16) play for $\Lambda(t)$.

Note also that we can look at the function $\varphi(t)$ as at the coordinate representation of some curve in $\mathbb{RP}^1 = L(W)$ with $\dim W = 2$. Therefore, all constructions and formulas of the previous section can be applied to this case. We denote by $g_\varphi(t_0,t_1)$ the function defined by (4.11), (4.14) with S_t replaced by $\varphi(t)$. Then differentiating both sides of (5.1) once w.r.t. t_0 and twice w.r.t. t_1, we obtain

$$\bar{g}\left(\varphi(t_0),\varphi(t_1)\right)\varphi'(t_0)\varphi'(t_1) = g(t_0,t_1) - kg_\varphi(t_0,t_1). \tag{5.2}$$

By (4.16) and (4.28) it follows that the substitution of $t_0 = t_1 = t$ into (5.2) gives the following reparametrization rule for the Ricci curvature:

$$\bar{\rho}(\tau)(\varphi'(t))^2 = \rho(t) - \frac{k}{3}\,\mathbb{S}(\varphi(t)). \tag{5.3}$$

Now we would like to find all reparametrizations $\tau = \varphi(t)$ such that the Ricci curvature $\bar{\rho}(\tau)$ in the new parameter τ is identically equal to zero.

The reparametrization rule (5.3) implies that such reparametrization have to satisfy the following differential equation:

$$\mathbb{S}\left(\varphi(t)\right) = \frac{3\rho(t)}{k}. \tag{5.4}$$

This equation has a solution at least locally (i.e., in a neighborhood of any given point) and as was already mentioned in Sec. 3, any two solutions are transformed one into another by a Möbius transformation. In other words, the set of all parametrizations of $\Lambda(\cdot)$ with the Ricci curvature identically equal to zero defines a projective structure on $\Lambda(\cdot)$ (any two parametrizations from this set are transformed one into another by a Möbius transformation). It is said to be the *canonical projective structure* of the curve $\Lambda(\cdot)$. The parameters of the canonical projective structure are said to be *projective parameters*.

Now we give a construction of a special form on a nonparametrized curve $\Lambda(\cdot)$ (namely, the fourth-order differential on $\Lambda(\cdot)$), which is the first appearing invariant of the nonparametrized curve. We will call it the fundamental form of the curve $\Lambda(\cdot)$.

Let t be a projective parameter on $\Lambda(\cdot)$. Then by definition $\rho(t) \equiv 0$, and by (4.18), $\beta_{0,1}(t) \equiv \frac{1}{2}\beta'_{0,0}(t) \equiv 0$. Therefore, by (4.26) we obtain that in the projective parameter,

$$g(t_0, t_1) = \beta_{0,2}(t_0)(t_1 - t_0)^2 + O\left((t_1 - t_0)^3\right). \tag{5.5}$$

Let τ be another projective parameter on $\Lambda(\cdot)$, i.e., $\tau = \varphi(t) = \dfrac{at+b}{ct+d}$. Then by Proposition 2, $g_\varphi(t_0, t_1) \equiv 0$. Substituting this into (5.2), we have

$$\bar{g}\left(\varphi(t_0), \varphi(t_1)\right)\varphi'(t_0)\varphi'(t_1) = g(t_0, t_1), \tag{5.6}$$

where $\tau_i = \varphi(t_i)$, $i = 0, 1$. Using (5.5), we compare the coefficients of the first terms in the Taylor expansions of both sides of (5.6). As a result, we obtain

$$\bar{\beta}_{0,2}(\varphi(t_0))(\varphi'(t_0))^4 = \beta_{0,2}(t_0)$$

or

$$\bar{\beta}_{0,2}(\tau)(d\tau)^4 = \beta_{0,2}(t)(dt)^4. \tag{5.7}$$

This means that the form $\beta_{0,2}(t)(dt)^4$ does not depend on the choice of the projective parameter t. This form is said to be a *fundamental form* of the curve $\Lambda(\cdot)$ and is denoted by $\mathcal{A}$.

If t is an arbitrary (not necessarily projective) parameter on the curve $\Lambda(\cdot)$, then the fundamental form $\mathcal{A}$ in this parameter should have the form $A(t)(dt)^4$, where $A(t)$ is a smooth function (the "density" of the fundamental form).

Lemma 5.1. *For an arbitrary parameter t, the density $A(t)$ of the fundamental form satisfies the relation*

$$A(t) = \beta_{0,2}(t) - \frac{3}{5k}\rho(t)^2 - \frac{3}{20}\rho''(t) \tag{5.8}$$

or, equivalently,

$$A(t) = \left(\frac{1}{10}\left(\frac{\partial}{\partial t_0} + \frac{\partial}{\partial t_1}\right)^2 - \frac{1}{2}\frac{\partial^2}{\partial t_0 \partial t_1}\right) g(t_0, t_1)\Big|_{t_0=t_1=t} - \\ - \frac{3}{5k} g(t,t)^2. \tag{5.9}$$

Proof. Let $\tau = \varphi(t)$ be a reparametrization such that τ is a projective parameter. It means that $\varphi(\tau)$ satisfies Eq. (5.4). Denote by $\beta^{\varphi}_{i,j}(t_0, t_1)$ the coefficients defined by (4.11), (4.14), and (4.16) with S_t replaced by $\varphi(t)$.

Using (5.5), compare the coefficients of the first terms in the Taylor expansions of both sides of (5.6). As a result, we obtain

$$\bar\beta_{0,2}(\varphi(t_0))(\varphi'(t_0))^4 = \beta_{0,2}(t_0) - k\beta^{\varphi}_{0,2}(t_0)$$

or

$$\mathcal{A} = \bar\beta_{0,2}(\tau)(d\tau)^4 = \left(\beta_{0,2}(t) - k\beta^{\varphi}_{0,2}(t)\right)(dt)^4. \tag{5.10}$$

To complete the proof, it remains to compute the coefficient $\beta^{\varphi}_{0,2}(t_0)$. For this we will use the recursive formula (4.30), where B_i are defined by (4.29) with $\beta^{\varphi}_{0,i}$ instead of $\beta_{0,2}$. From (4.30) it follows that

$$B_2(t) = \frac{1}{4}B_1'(t),$$
$$B_3(t) = \frac{1}{5}\left(B_2'(t) - (B_1(t))^2\right) = \frac{1}{20}B_1''(t) - \frac{1}{5}(B_1(t))^2. \tag{5.11}$$

From (4.28), (4.29), and (5.4) it follows that

$$B_1(t) = -\beta^{\varphi}_{0,0}(t) = -\frac{1}{3}\mathbb{S}\left(\varphi(t)\right) = -\frac{\rho(t)}{k}.$$

Then by (4.30) and (5.11),

$$\beta^{\varphi}_{0,2}(t) = -3B_3(t) = \frac{3}{20k}\rho''(t) + \frac{3}{5k^2}\rho(t)^2.$$

This, together with (5.10), implies (5.8). To obtain (5.9), we rewrite (5.8), taking into account the connection between the function $g(t_0, t_1)$ and the functions $\rho(t)$ $(= \beta_{0,0}(t))$ and $\beta_{0,2}(t)$ given by expansion (4.16) (relation (5.9) is the most symmetric one w.r.t. t_0 and t_1). $\square$

If $A(t)$ does not change sign, then the canonical length element $|A(t)|^{\frac{1}{4}}dt$ is defined on $\Lambda(\cdot)$. The corresponding parameter τ (i.e., the length with respect to this length element) is called *a normal parameter* (in particular, this implies that abnormal extremals may have the canonical (normal) parametrization). Calculating the Ricci curvature $\rho_n(\tau)$ of $\Lambda(\cdot)$ in the normal parameter, we obtain a functional invariant of the nonparametrized curve. It is called *projective curvature* of the nonparametrized curve $\Lambda(\cdot)$. If $t=\varphi(\tau)$ is the transition function between a projective parameter t and the normal parameter τ, then by (5.4) it follows that $\rho_n(\tau)=\dfrac{k}{3}\mathbb{S}\left(\varphi(\tau)\right)$.

At the end of this section we give an explicit formula for the fundamental form of the regular curve in terms of its curvature operator. We note first that by definition the weight k of the regular curve is equal to $\dfrac{1}{2}\dim W$ (one can also derive it from (3.2) and (4.21)).

Lemma 5.2. *The fundamental form $\mathcal{A}$ of the regular curve $\Lambda(t)$ in the Lagrange Grassmannian $L(W)$ satisfies the relation*

$$\mathcal{A}=\frac{1}{15}\left(\mathrm{tr}\big(R(t)^2\big)-\frac{1}{k}\big(\mathrm{tr}R(t)\big)^2\right)(dt)^4, \tag{5.12}$$

where $R(t)$ is the curvature operator of $\Lambda(t)$ defined by (3.1) *and* $k=\dfrac{1}{2}\dim W$.

Proof. Let us compute $\beta_{0,2}(t)$. We will use the notation of (2.2) and (2.3). By (4.22),

$$\beta_{0,2}(t)=-3\mathrm{tr}(Q_3(t)\dot{\Lambda}(t))=-3\mathrm{tr}(A_3(t)\dot{S}(t)). \tag{5.13}$$

For a given $\bar{t}$, we choose for simplicity a coordinate representation S_t of the curve $\Lambda(t)$ such that $A_0(\bar{t})=0$. Then by (3.3),

$$A_3(\bar{t})=\frac{1}{5}\big(\dot{A}_2(\bar{t})-A_1(\bar{t})\dot{S}_{\bar{t}}A_1(\bar{t})\big). \tag{5.14}$$

From (3.4) it follows that the condition $A_0(\bar{t})=0$ is equivalent to $\dddot{S}_{\bar{t}}=0$. This implies that $\dot{A}_2(\bar{t})\dot{S}_{\bar{t}}=\dfrac{d}{dt}(A_2(t)\dot{S}_t)\Big|_{t=\bar{t}}$. Therefore, multiplying (5.14) by $S_{\bar{t}}$ and taking the trace from both sides, we obtain

$$\mathrm{tr}\big(A_3(\bar{t})\dot{S}_{\bar{t}}\big)=\frac{1}{5}\frac{d}{dt}\mathrm{tr}\big(A_2(\bar{t})\dot{S}_{\bar{t}}\big)-\frac{1}{5}\mathrm{tr}\big((A_1(\bar{t})\dot{S}_{\bar{t}})^2\big). \tag{5.15}$$

Now by (4.22) and (4.18),

$$\mathrm{tr}\left(A_2(\bar{t})\dot{S}_{\bar{t}}\right)=-\frac{1}{2}\beta_{0,1}(\bar{t})=-\frac{1}{4}\rho'(\bar{t}). \tag{5.16}$$

On the other hand, by (3.6) $(A_1(\bar t)\dot S_{\bar t}) = -\dfrac{1}{3}R(\bar t)$. This and (5.16) imply that (5.15) can be written in the form

$$\operatorname{tr}\big(A_3(\bar t)\dot S_{\bar t}\big) = -\frac{1}{20}\rho''(\bar t) - \frac{1}{45}\operatorname{tr}\big(R(\bar t)^2\big)\,.$$

Taking into account (5.14), we obtain by (5.8) that

$$\begin{aligned} A(\bar t) &= \frac{3}{20}\rho''(\bar t) + \frac{1}{15}\operatorname{tr}\big(R(\bar t)^2\big) - \frac{3}{5k}\rho(\bar t)^2 - \frac{3}{20}\rho''(\bar t) = \\ &= \frac{1}{15}\operatorname{tr}\big(R(\bar t)^2\big) - \frac{3}{5k}\rho(\bar t)^2. \end{aligned}$$

Finally, note that $\rho = \dfrac{1}{3}\operatorname{tr} R$ (see (4.27)). Substituting this into the last relation, we obtain (5.12). $\square$

Note that in the scalar case (i.e., when $\dim W = 2$) the fundamental form $\mathcal{A}$ is identically equal to zero.

Remark 2. All constructions of Secs. 3–5 can be made for the curve in the Grassmannian $G(m, 2m)$ (the set of all m-dimensional subspaces in the $2m$-dimensional linear space) instead of the Lagrangian Grassmannian by the action of the group $\mathrm{GL}(2m)$ instead of the symplectic group.

6. The rank 1 curves: preliminary steps.

In the present section we begin a systematic study of the curves of rank 1 in the Lagrange Grassmannian $L(W)$ with $\dim W = m$. We consider a rank 1 ample curve $\Lambda : I \mapsto L(W)$ with, possibly, a nonconstant weight, where I is some interval on the real line. We introduce a canonical basis on each subspace $\Lambda(t)$ and compute some characteristics of the curve, in particular, its weight at any point. Finally, we show that the curve Λ has the constant weight equal to m^2 on the set with a discrete complement in I. All this will prepare us to the next section, where the curves of rank 1 and constant weight will be investigated.

Without loss of generality, suppose that $\Lambda(\tau)$ is monotone ally nondecreasing, i.e., the velocities $\dot\Lambda(t)$ are nonnegtive definite quadratic forms. As in Sec. 2, let $\Lambda_\tau(t)$ be the identical imbedding of $\Lambda(t)$ into the affine space $\Lambda(\tau)^{\pitchfork}$. The velocity $\dfrac{\partial}{\partial t}\Lambda_\tau(t)$ is a well-defined self-adjoint linear mapping from $\Lambda(\tau)^*$ to $\Lambda(\tau)$, i.e., an element of $\mathrm{Sym}^2\Lambda(\tau)$. Moreover, by our assumptions, $\dfrac{\partial}{\partial t}\Lambda_\tau(t)$ is a nonpositive self-adjoint linear mapping of rank 1. Therefore, for $t \neq \tau$ there exists a unique, up to the sign, vector $w(t,\tau) \in \Lambda(\tau)$

such that for any $v \in \Lambda(\tau)^*$

$$\left\langle v, \frac{\partial}{\partial t}\Lambda_\tau(t)v\right\rangle = -\langle v, w(t,\tau)\rangle^2. \tag{6.1}$$

Remark 3. From the definition of $w(t,\tau)$ it easily follows that for given τ the germ of the curve $\Lambda(t)$ at $t=\tau$ is defined uniquely by $\Lambda(\tau)$, the derivative subspace $\Lambda^0(\tau)$, and the germ of the function $t \mapsto w(t,\tau)$ at $t=\tau$. Since the symplectic group acts transitively on the set of pairs of transversal Lagrange subspaces, one can conclude that the germ of the curve $\Lambda(t)$ at $t=\tau$ is defined uniquely, up to a symplectic transformation, by the germ of the function $t \mapsto w(t,\tau)$ at $t=\tau$.

The function $t \mapsto \Lambda_\tau(t)$ has a pole at $t=\tau$. This easily implies that the function $t \mapsto w(t,\tau)$ also has a pole at $t=\tau$. Suppose that the order of this pole is equal to $l(\tau)$.

Denote by $u(t,\tau)$ the normalized curve $t \to u(t,\tau) = (t-\tau)^{l(\tau)}w(t,\tau)$ and define the following vectors in $\Lambda(\tau)$:

$$e_j(\tau) = \frac{1}{(j-1)!}\,\frac{\partial^{j-1}}{\partial t^{j-1}}u(t,\tau)\bigg|_{t=\tau}. \tag{6.2}$$

We note first that

$$\operatorname{span}\left(\{e_j(\tau)\}_{j=1}^{\infty}\right) = \Lambda(\tau). \tag{6.3}$$

Otherwise, using the relation

$$w(t,\tau) = \sum_{i=1}^{j} e_i(\tau)(t-\tau)^{i-1-l} + O((t-\tau)^{i-l}),$$

one can easily obtain the contradiction to the fact that $\Lambda(t)$ is ample.

Thus for a given parameter τ and integer i, $1 \le i \le m$, the following integers $k_i(\tau)$ are well defined:

$$\begin{aligned} k_i(\tau) = \min\bigl\{&j \in \mathbb{N}\cup 0 : \\ &\dim\left(\operatorname{span}\left(e_1(\tau), e_2(\tau), \dots, e_{j+1}(\tau)\right)\right) = i\bigr\}. \end{aligned} \tag{6.4}$$

Note that

$$0 = k_1(\tau) < k_2(\tau) < \ldots < k_m(\tau), \quad k_i(\tau) \ge i-1. \tag{6.5}$$

By definition, the vectors $e_{k_1(\tau)+1}(\tau), \dots, e_{k_m(\tau)+1}(\tau)$ constitute the basis of the subspace $\Lambda(\tau)$. This basis is said to be a *canonical basis* of $\Lambda(\tau)$. Since the vector $w(t,\tau)$ is defined up to the sign, the vector $e_1(\tau)$ $(= e_{k_1(\tau)+1}(\tau))$ is also defined up to the sign. Therefore, one can take also

$$(-e_{k_1(\tau)+1}(\tau), \dots, -e_{k_m(\tau)+1}(\tau))$$

as a canonical basis on the plane $\Lambda(\tau)$. Denote by $w_i(t,\tau)$ the ith component of the vector $w(t,\tau)$ w.r.t. this basis. In other words, functions $w_i(t,\tau)$ satisfy the relation

$$w(t,\tau) = \sum_{i=1}^{m} w_i(t,\tau) e_{k_i(\tau)+1}(\tau). \tag{6.6}$$

Remark 4. Using Remark 3, one can easily conclude that the germ of the curve $\Lambda(t)$ at $t=\tau$ is defined uniquely by $\Lambda(\tau)$, the canonical basis in $\Lambda(\tau)$, the derivative subspace $\Lambda^0(\tau)$, and the germs of the functions $t \mapsto w_i(t,\tau)$ at $t=\tau$, where $1 \le i \le m$. Since for any two pairs (Λ,Δ) and $(\tilde\Lambda,\tilde\Delta)$ of transversal Lagrange subspaces with fixed bases in Λ and $\tilde\Lambda$ there exists a symplectic transformation that transforms basis in Λ into the basis in $\tilde\Lambda$ and subspace Δ into $\tilde\Delta$, we have that the germ of the curve $\Lambda(t)$ at $t=\tau$ is defined uniquely, up to a symplectic transformation, by the germ of the functions $t \mapsto w_i(t,\tau)$ at $t=\tau$, where $1 \le i \le m$.

Now we prove a computational lemma about the weight of $\Lambda(t)$ at τ and the order of the pole of $t \mapsto w(t,\tau)$.

Lemma 6.1. *The order $l(\tau)$ of the pole of the function $t \mapsto w(t,\tau)$ is equal to $k_m(\tau)+1$. The weight of the curve $\Lambda(t)$ at τ is equal to $(2\,k_m(\tau)+1)\,m - 2\sum_{i=2}^{m} k_i(\tau)$.*

Proof. For simplicity we will write k_i instead of $k_i(\tau)$ and l instead of $l(\tau)$. Let S_t, $\Lambda(t) = \{(x, S_t\,x) : x \in \mathbb{R}^m\}$, be a coordinate representation of the germ of $\Lambda(t)$ at $t=\tau$ such that the canonical basis $e_{k_1+1}(\tau), \dots, e_{k_m+1}(\tau)$ is a standard basis of $\mathbb{R}^m$. Denote the subspace $D \in \Lambda(\tau)^{\pitchfork}$ by $\Delta = 0 \oplus \mathbb{R}^m$. From (6.2) it follows that

$$w_i(t,\tau) = (t-\tau)^{k_i-l} + O((t-\tau)^{k_i-l+1}) \tag{6.7}$$

in the canonical basis.

Then relation (6.1) in the canonical basis can be rewritten in the form

$$\left(\frac{\partial}{\partial t}(S_t - S_\tau)^{-1}\right)_{i,j} = -w_i(t,\tau)w_j(t,\tau) = -(t-\tau)^{k_i+k_j-2l} + \\ + O((t-\tau)^{k_i+k_j-2l+1}). \tag{6.8}$$

For simplicity take coordinates $t \mapsto S_t$ such that the subspace Δ is the derivative subspace $\Lambda^0(\tau)$. Then by definition of the derivative subspace, the free term on the Laurent expansion of $(S_t - S_\tau)^{-1}$ is equal to zero. Therefore,

$$\Big((S_t - S_\tau)^{-1}\Big)_{i,j} = -\int^{t} w_i(\xi,\tau)w_j(\xi,\tau)\,d\xi = \frac{(t-\tau)^{2l-k_i-k_j-1}}{k_i+k_j-2l+1} + \\ + O((t-\tau)^{k_i+k_j-2l+2}). \tag{6.9}$$

Then it is easy to obtain the following expansion for the determinant:

$$\det(S_t - S_\tau) = \frac{(t-\tau)^k}{C} + O((t-\tau)^{k+1}), \tag{6.10}$$

where

$$k = (2\,l-1)\,m - 2\sum_{i=2}^{m} k_i \tag{6.11}$$

and C is the determinant of the matrix whose (i,j)th entry is $\dfrac{1}{2l-k_i-k_j-1}$, $i,j = 1,\dots,m$. It is well known that the determinant of the matrix whose (i,j)th entry is $\dfrac{1}{x_i+y_j}$, $i,j = 1,\dots,m$, can be computed by the formula

$$\det\left(\left\{\frac{1}{x_i+y_j}\right\}_{i,j=1}^{m}\right) = \frac{\prod\limits_{1\le i<j\le m}(x_i-x_j)(y_i-y_j)}{\prod\limits_{i,j=1}^{m}(x_i+y_j)}. \tag{6.12}$$

This implies in particular that $C \neq 0$ (one can take $x_i = y_i = l - k_i - \dfrac{1}{2}$ and use the fact that $k_i \neq k_j$ for $i \neq j$). Therefore, the weight is equal to $(2l-1)m - 2\sum\limits_{i=2}^{m} k_i$.

Further, from (6.9) and (6.12) it follows that

$$(S_t - S_\tau)_{i,j} = \big(C_{i,j}(t-\tau)^{-k+2l-k_i-k_j-1} + \\ + O((t-\tau)^{-k+2l-k_i-k_j})\big)\left(\frac{(t-\tau)^k}{C} + O((t-\tau)^{k+1})\right) = \\ = \frac{C_{i,j}}{C}(t-\tau)^{2l-k_i-k_j-1} + O((t-\tau)^{2l-k_i-k_j}), \tag{6.13}$$

where C is as in (6.10), k is as in (6.11), and $C_{i,j}$ are (i,j)th entries of the adjacent matrix to the matrix $\left(\dfrac{1}{2l-k_i-k_j-1}\right)_{i,j=1}^{m}$. By (6.12) and (6.5),

$C_{i,j} \neq 0$. Since S_t is a smooth curve at τ, all powers $2l - k_i - k_j - 1$ in (6.13) are positive. By assumption, $\dot{S}_\tau \neq 0$. It implies that

$$\min_{1 \le i,j \le m} (2l - k_i - k_j - 1) = 1. \tag{6.14}$$

But from (6.5) it follows that $\min\limits_{1 \le i,j \le m} (2l - k_i - k_j - 1) = 2l - 2k_m - 1$, which yields $l = k_m + 1$. Consequently, the weight is equal to $(2\,k_m + 1)\,m - 2\sum\limits_{i=2}^{m} k_i$. $\square$

Remark 5. In the proof of the previous lemma we have taken the coordinate representation $t \mapsto S_t$, $\Lambda(t) = \big\{(x, S_t\, x) : x \in \mathbb{R}^m\big\}$ with $\Delta = \Lambda^0(\tau)$ (where $\Delta = 0 \oplus \mathbb{R}^m$) in order to obtain asymptotics (6.9). But then we have obtained relation (6.14) which implies that $k_i(\tau) + k_j(\tau) - 2l(\tau) + 1 < 0$ for any $i, j = 1, \dots, m$. Therefore, asymptotics (6.9) for $\big((S_t - S_\tau)^{-1}\big)_{i,j}$ and, therefore, asymptotics (6.13) for $(S_t - S_\tau)_{i,j}$ are valid for any coordinate representation $t \mapsto S_t$, $\Lambda(t) = \big\{(x, S_t\, x) : x \in \mathbb{R}^m\big\}$, of the germ of $\Lambda(t)$ at $t = \tau$ such that the canonical basis $e_{k_1+1}(\tau), \dots, e_{k_m+1}(\tau)$ is a standard basis of $\mathbb{R}^m$ and $\Delta = 0 \oplus \mathbb{R}^m$ is an arbitrary subspace transversal to $\Lambda(\tau)$. The reason is that asymptotics (6.9) do not depend on the free term.

Take some subspace $\Delta \in \Lambda(\tau)^{\pitchfork}$. Recall that the velocity $\dot{\Lambda}(t)$ is a self-adjoint nonnegative definite linear mapping of rank 1 from $\Lambda(t)$ to $\Lambda(t)^*$. For any t sufficiently close to τ one can identify Δ with $\Lambda(t)^*$. Under this identification $\dot{\Lambda}(t)$ is a self-adjoint nonnegative linear mapping of rank 1 from $\Lambda(t)$ to Δ. Therefore, there exists a unique, up to sign, vector $v(t) \in \Delta$ such that for any $w \in \Lambda(t)$:

$$\langle \dot{\Lambda}(t) w, w \rangle = \langle v(t), w \rangle^2. \tag{6.15}$$

Suppose that a tuple of vectors $f_1(\tau), \dots, f_m(\tau)$ is a basis of Δ dual to the canonical basis of $\Lambda(\tau)$ (i.e., $\sigma(f_i(\tau), e_{k_j(\tau)+1}(\tau)) = \delta_{i,j}$). From Remark 5 and relation (6.13) (where $l = k_m + 1$) it follows that the components $v_i(t)$ of the vector $v(t)$ w.r.t. the basis $f_1(\tau), \dots, f_m(\tau)$ have the following asymptotics:

$$v_i(t) = c_i(\tau)(t - \tau)^{k_m(\tau) - k_i(\tau)} + O\left((t - \tau)^{k_m(\tau) - k_i(\tau) + 1}\right), \tag{6.16}$$

where $c_i(\tau) \neq 0$. (Actually, using (6.12), one can compute $c_i(\tau)$:

$$c_i(\tau) = \sqrt{\frac{C_{i,i}(\tau)(2(k_m(\tau) - k_i(\tau)) + 1)}{C(\tau)}} = \\ = \frac{\prod_{j=1}^{m} (2k_m(\tau) - k_i(\tau) - k_j(\tau) + 1)}{\prod_{1 \le j \le m,\, j \neq i} (k_i(\tau) - k_j(\tau))}, \tag{6.17}$$

where $C_{i,i}$ and C are as in the proof of Lemma 6.1.) Relation (6.16) implies that the relation

$$\operatorname{span}(v(\tau), v'(\tau), \dots, v^{(j)}(\tau)) = \operatorname{span}(f_i(\tau), \dots, f_m(\tau)) \tag{6.18}$$

holds for any integer nonnegative j such that

$$k_m(\tau) - k_i(\tau) \le j < k_m(\tau) - k_{i-1}(\tau). \tag{6.19}$$

In particular,

$$\operatorname{span}(v(\tau), v'(\tau), \dots, v^{(k_m(\tau)-1)}(\tau)) = \\ = \operatorname{span}(f_2(\tau), \dots, f_m(\tau)) \subsetneq \Delta, \tag{6.20}$$

$$\operatorname{span}(v(\tau), v'(\tau), \dots, v^{(k_m(\tau))}(\tau)) = \Delta \tag{6.21}$$

(recall that $k_1(\tau) = 0$).

Now we are ready to prove the following proposition

Proposition 3. *For the ample curve $\Lambda : I \mapsto L(W)$ of rank 1 the set $\mathcal{C}$ such that*

$$\mathcal{C} = \{t \in I : \dim(\operatorname{span}(e_1(t), e_2(t), \dots, e_m(t))) < m\} \tag{6.22}$$

is a discrete set of the interval of definition I.

Proof. Suppose that $\mathcal{C}$ has an accumulation point τ. Take some subspace $\Delta \in \Lambda(\tau)^{\pitchfork}$. Let $t \mapsto v(t)$ be a curve of vectors in Δ defined by (6.15) for all t from some neighborhood U of τ in I. Note that $t \in \mathcal{C}$ iff $k_m(t) \ge m$. Therefore, by (6.20) and (6.21) we have that $t_0 \in \mathcal{C} \cap U$ iff the function $d(t) \stackrel{\text{def}}{=} \det(v(t), v'(t), \dots v^{(m-1)}(t))$ has a zero at $t = t_0$. For the accumulation point τ, using consequently the Rolle theorem, one can conclude that the function $d(t)$ has a zero of infinite order at $t = \tau$.

On the other hand, let $l_i(\tau) \stackrel{\text{def}}{=} k_m(\tau) - k_{m-i+1}(\tau)$. Denote $p = \sum_{i=1}^{m} l_i(\tau)$. Let us prove that $d^{(p)}(\tau)$ is not equal to zero. Indeed, $d^{(p)}(\tau)$ can be expressed as a sum of the terms of the form $\det(v^{(j_1)}(\tau), \dots, v^{(j_m)}(\tau))$, where

$$\sum_{i=1}^{m} j_i = p, \quad 0 \le j_1 < j_2 < \dots < j_m. \tag{6.23}$$

Let us show that if the tuple $(j_1, \dots, j_m)$ is different from the tuple $(l_1(\tau), \dots, l_m(\tau))$ and satisfies (6.23), then

$$\det(v^{(j_1)}(\tau), \dots, v^{(j_m)}(\tau)) = 0. \tag{6.24}$$

For this, we first note that by assumptions there exists an index s such that $j_s < l_s(\tau)$ $(= k_m(\tau) - k_{m-s+1}(\tau))$. Then we have from (6.19) and (6.18)

$$\operatorname{span}(v^{(j_1)}(\tau), \dots, v^{(j_s)}(\tau)) \subset \operatorname{span}(f_{m-s+2}(\tau), \dots, f_m(\tau)),$$

i.e., $\dim\big(\operatorname{span}(v^{(j_1)}(\tau), \dots, v^{(j_s)}(\tau))\big) < s$. This implies that $\dim\big(\operatorname{span}(v^{(j_1)}(\tau), \dots, v^{(j_m)}(\tau))\big) < m$ which is equivalent to (6.24). Also, we note that it easily follows from (6.19) and (6.18) that $\operatorname{span}\big(v^{(l_1(\tau))}(\tau), \dots, v^{(l_m(\tau))}(\tau)\big) = \Delta$. Therefore,

$$d^{(p)}(\tau) = c \det(v^{(l_1(\tau))}(\tau), \dots, v^{(l_m(\tau))}(\tau)) \neq 0$$

(here c is some natural number). Hence $d(t)$ has a zero of finite order at $t = \tau$. We obtain the contradiction. $\square$

For $t \in I \backslash \mathcal{C}$, the numbers $k_i(t) = i - 1$. As a consequence of the previous proposition and the relation for the weight from Lemma 6.1, we obtain the following corollary.

Corollary 1. *The ample curve $\Lambda : I \mapsto L(W)$ of rank 1 has the constant weight equal to m^2 on the set with a discrete complement in I.*

At the end of this section we give an explicit formula for the velocity $\dot\Lambda(\tau)$ in the canonical basis. Let $(e_1^*(\tau), \dots, e_m^*(\tau))$ be a basis in $\Lambda(\tau)^*$ dual to the canonical basis in $\Lambda(\tau)$. As we have seen at the end of the proof of Lemma 6.1, the (m, m)th entry is the only nonzero entry of the matrix $\dot S_\tau$ and is equal to $\dfrac{C_{m,m}(\tau)}{C(\tau)} = c_m^2(\tau)$ (where $c_m(\tau)$ is as in (6.17)). Therefore, we obtain the following lemma.

Lemma 6.2. *For any $v_1, v_2 \in \Lambda(\tau)$ the following relation holds*:

$$\langle \dot\Lambda(\tau) v_1, v_2 \rangle = c_m^2(\tau) \langle e_m^*(\tau), v_1 \rangle \langle e_m^*(\tau), v_2 \rangle, \tag{6.25}$$

where

$$c_m(\tau) = \prod_{j=1}^{m-1} \frac{k_m(\tau) - k_j(\tau) + 1}{k_m(\tau) - k_j(\tau)}. \tag{6.26}$$

7. The rank 1 curves with a constant weight

This section is devoted to the curves of rank 1 and a constant finite weight in the Lagrange Grassmannian $L(W)$. We show that in this case the function $g(t,\tau)$ constructed in Sec. 4 defines the curve uniquely, up to a symplectic transformation. We also find a complete system of invariants of the curve in terms of the function g.

First, using Proposition 3 and Lemma 6.1, we obtain the following proposition.

Proposition 4. *If $\Lambda(t)$ is a curve of rank 1 and a constant weight on I, then for all $t \in I$ and $1 \le i \le m$ the numbers $k_i(t)$ are equal to $i-1$, or, equivalently, the vectors $e_1(t), \dots, e_m(t)$ constitute the canonical basis of the subspace $\Lambda(t)$.*

Proof. From (6.5) it follows that always

$$k_i(t) - k_j(t) \ge i - j, \quad k_1(t) = 0. \tag{7.1}$$

Therefore, by Lemma 6.1 the weight $k(t)$ of the curve Λ at the point t satisfies the relation

$$\begin{aligned} k(t) &= (2k_m(t)+1)m - 2\sum_{i=2}^{m} k_i(t) = \\ &= 2\sum_{i=1}^{m}(k_m(t) - k_i(t)) + m \ge 2\sum_{i=1}^{m}(m-i) + m = m^2. \end{aligned} \tag{7.2}$$

In addition, from (7.1) it is easy to see that the equality in (7.2) holds iff $k_i(t) = i-1$ for any $1 \le i \le m$. Therefore, if the set $\mathcal{C}$ is as in (6.22), then for any $t \in \mathcal{C}$ the weight $k(t) > m^2$, while for $t \notin \mathcal{C}$ the weight $k(t) = m^2$. But by Proposition 3 the set $\mathcal{C}$ is a discrete subset of I. Hence, the set $\mathcal{C}$ should be empty for the weight $k(t)$ to be constant on I. This completes the proof of the proposition. □

As a consequence of the previous proposition and Lemmas 6.1 and 6.2, we easily obtain the following corollary.

Corollary 2. *If $\Lambda(t)$ is a curve of rank* 1 *and a constant weight on I, then*:

(1) *the weight is equal to m^2 at any point $t \in I$*;

(2) *for any $\tau \in I$ the function $t \mapsto w(t,\tau)$ has a pole of order m at $t = \tau$*;

(3) *for any* $v_1, v_2 \in \Lambda(\tau)$ *the following relation holds*:

$$\langle \dot{\Lambda}(\tau) v_1, v_2 \rangle = m^2 \langle e_m^*(\tau), v_1 \rangle \langle e_m^*(\tau), v_2 \rangle. \tag{7.3}$$

Now we prove that the function $g(t, \tau)$ defined in Sec. 4 contains all information about $\Lambda(t)$.

Theorem 1. *The function* $g(t, \tau)$ *defines the curve* $\Lambda(t)$ *of rank* 1 *and a constant weight uniquely, up to a symplectic transformation.*

Before starting to prove the theorem, we want to describe in few words the main steps of the proof. First, we show that the function $g(t, \tau)$ is almost the same as the component $w_m(t, \tau)$ of the vector $w(t, \tau)$. The vector $w(t, \tau)$ is a function of two variables, but it is defined by a curve. Therefore, it is natural to expect that $w(t, \tau)$ satisfies some partial differential equation. We find this equation that is actually the system of m equations for the components $w_i(t, \tau)$, $1 \le i \le m$. Then we show that this system has a "triangular" form such that all components $w_i(t, \tau)$ can be expressed in terms of $w_m(t, \tau)$ and refer to Remark 4 to complete the proof.

7.1. Proof of Theorem 1.

1. We begin the proof with the following lemma.

Lemma 7.1. *The following relation holds*:

$$w_m^2(t, \tau) = \frac{1}{(t-\tau)^2} + \frac{1}{m^2} g(t, \tau). \tag{7.4}$$

Proof. By (4.20), (4.23), and item (2) of Corollary 4 we have

$$\operatorname{tr}\left(\frac{\partial}{\partial t} \Lambda_\tau(t) \circ \dot{\Lambda}(\tau) \right) = -\frac{m^2}{(t-\tau)^2} - g(t, \tau). \tag{7.5}$$

Let $t \to S_t$, $\Lambda(t) = \{(x, S_t\, x) : x \in \mathbb{R}^m\}$ be a coordinate representation of the germ of $\Lambda(t)$ at $t = \tau$ such that the canonical basis $e_1(\tau), \ldots, e_m(\tau)$ is a standard basis of $\mathbb{R}^m$. By (4.24),

$$\operatorname{tr}\left(\frac{\partial}{\partial t} \left((S_t - S_\tau)^{-1} \right) \dot{S}_\tau \right) = -\frac{m^2}{(t-\tau)^2} - g(t, \tau). \tag{7.6}$$

Relation (7.3) implies that in the chosen coordinates

$$\dot{S}_\tau = \begin{cases} 0, & (i, j) \ne (m, m), \\ m^2, & (i, j) = m. \end{cases}$$

By construction,

$$\left(\frac{\partial}{\partial t} \left((S_t - S_\tau)^{-1} \right) \right)_{i,j} = -w_i(t, \tau) w_j(t, \tau).$$

Therefore,

$$\operatorname{tr}\left(\frac{\partial}{\partial t}\left((S_t - S_\tau)^{-1}\right)\dot{S}_\tau\right) = -m^2 w_m^2(t,\tau).$$

This together with (7.6) implies (7.4). $\square$

By (7.4) it follows that in order to prove the theorem it is sufficient to show that the function $w_m(t,\tau)$ defines $\Lambda(t)$ uniquely, up to a symplectic transformation.

2. Now we derive a partial differential equation for the vector function $w(t,\tau)$.

Lemma 7.2. *The vector function $w(t,\tau)$ satisfies the following differential equation*:

$$\frac{\partial^2 w}{\partial t \partial \tau} - \left(\frac{\frac{\partial w_m}{\partial t}}{w_m}\right)\frac{\partial w}{\partial \tau} + m^2 w_m^2 w = 0. \tag{7.7}$$

Proof. We fix some parameter τ_0 and take some subspace Δ transversal to $\Lambda(\tau_0)$. Let $t \to S_t$, $\Lambda(t) = \{(x, S_t x) : x \in \mathbb{R}^m\}$ be a coordinate representation of the germ of $\Lambda(t)$ at $t = \tau_0$ such that $\Lambda(\tau_0) = \mathbb{R}^m \oplus 0$ and $\Delta = 0 \oplus \mathbb{R}^m$. Denote by $w^\Delta(t,\tau) \in \mathbb{R}^m$ the first m components of the vector $w(t,\tau)$ in the chosen coordinates (or equivalently, the image of $w(t,\tau)$ under the projection of W on $\Lambda(\tau_0)$ parallel to Δ). Also, let, as before, $f_1(\tau), \dots, f_m(\tau)$ be the basis of Δ dual to the canonical basis of $\Lambda(\tau)$ (w.r.t. the symplectic form σ).

By (6.1), it follows that for t and τ close to τ_0

$$\frac{\partial}{\partial t}\left((S_t - S_\tau)^{-1}\right) = -w^\Delta(t,\tau) w^\Delta(t,\tau)^T. \tag{7.8}$$

Therefore,

$$\dot{S}_t = \left((S_t - S_\tau) w^\Delta(t,\tau)\right)\left((S_t - S_\tau) w^\Delta(t,\tau)\right)^T. \tag{7.9}$$

This implies that the vector function $(S_t - S_\tau) w^\Delta(t,\tau)$ does not depend on τ. Differentiating it w.r.t. τ, we obtain

$$-\dot{S}_\tau w^\Delta(t,\tau) + (S_t - S_\tau)\frac{\partial}{\partial \tau} w^\Delta(t,\tau) = 0. \tag{7.10}$$

From (6.25) it follows that

$$\dot{S}_\tau w^\Delta(t,\tau) = m^2 w_m(t,\tau) f_m(\tau). \tag{7.11}$$

This together with (7.10) implies that

$$\frac{\partial}{\partial \tau} w^\Delta(t,\tau) = m^2 w_m(t,\tau)(S_t - S_\tau)^{-1} f_m(\tau). \tag{7.12}$$

In particular,

$$m^2(S_t - S_\tau)^{-1} f_m(\tau) = \frac{1}{w_m(t,\tau)} \frac{\partial}{\partial \tau} w^\Delta(t,\tau). \tag{7.13}$$

Now, differentiating (7.12) w.r.t. t, we have

$$\begin{aligned}\frac{\partial^2}{\partial t \partial \tau} w^\Delta(t,\tau) = m^2 w_m(t,\tau) \frac{\partial}{\partial t}\Big((S_t - S_\tau)^{-1}\Big) f_m(\tau) + \\ + \frac{\partial}{\partial t} w_m(t,\tau) m^2 (S_t - S_\tau)^{-1} f_m(\tau).\end{aligned} \tag{7.14}$$

From (7.8) it follows that

$$(S_t - S_\tau)^{-1} f_m(\tau) = -w_m(t,\tau) w^\Delta(t,\tau).$$

Substituting this and (7.13) into (7.14), we obtain

$$\frac{\partial^2 w^\Delta}{\partial t \partial \tau} - \left(\frac{\frac{\partial w_m}{\partial t}}{w_m}\right) \frac{\partial w^\Delta}{\partial \tau} + m^2 w_m^2 w^\Delta = 0. \tag{7.15}$$

Recalling the definition of $w^\Delta(t,\tau)$, we obtain from the last equation the following inclusion:

$$\frac{\partial^2 w}{\partial t \partial \tau} - \left(\frac{\frac{\partial w_m}{\partial t}}{w_m}\right) \frac{\partial w}{\partial \tau} + m^2 w_m^2 w \in \Delta. \tag{7.16}$$

Recall that all our considerations (and, in particular, inclusion (7.16)) are valid for any subspace Δ transversal to $\Lambda(\tau_0)$ and any t and τ close to τ_0. Taking as Δ in (7.16) two subspaces that are transversal to $\Lambda(\tau_0)$ and also transversal one to another, we obtain (7.7) for any t and τ close to τ_0. Since τ_0 is arbitrary, this completes the proof of the lemma. $\square$

In the sequel it is also convenient to make the following substitution in (7.7):

$$Y(t,\tau) = \frac{1}{w_m(t,\tau)} w(t,\tau). \tag{7.17}$$

Then by a direct computation one can obtain the following equation for Y:

$$\frac{\partial^2 Y}{\partial t \partial \tau} + \left(\frac{\frac{\partial w_m}{\partial \tau}}{w_m}\right) \frac{\partial Y}{\partial t} + \left(\frac{\partial^2}{\partial t \partial \tau}(\ln w_m) + m^2 w_m^2\right) Y = 0. \tag{7.18}$$

3. Now we rewrite Eq. (7.7) as a system of equations w.r.t. the components $w_i(t,\tau)$. Take some subspace $\Delta \in \Lambda(\tau)^{\pitchfork}$. Identifying Δ with $\Lambda(\tau)^*$, denote by $f_i(\tau)$ the vector, corresponding to $e_i^*(\tau)$ under this identification.

The vectors $e_1(\tau),\dots,e_m(\tau), f_1(\tau),\dots,f_m(\tau)$ constitute the basis of the symplectic space W. Suppose that

$$\dot e_i(\tau)=\sum_{j=1}^m \alpha_{i,j}(\tau)e_j(\tau)+\gamma_{i,j}(\tau)f_j(\tau).$$

According to (7.3),

$$\gamma_{ij}(\tau)=\begin{cases}0, & (i,j)\neq(m,m),\\ m^2, & (i,j)=(m,m).\end{cases}$$

This implies that

$$\begin{aligned}\dot e_i(\tau)&=\sum_{j=1}^m \alpha_{i,j}(\tau)e_j(\tau),\ 1\le i\le m-1,\\ \dot e_m(\tau)&=\sum_{j=1}^m \alpha_{m,j}(\tau)e_j(\tau)+m^2 f_m(\tau).\end{aligned}\tag{7.19}$$

Remark 6. In particular, it follows that the functions $\alpha_{i,j}(\tau)$ with $1\le i\le m-1$ do not depend on the choice of the subspace Δ.

By definition,

$$w(t,\tau)=\sum_{i=1}^m w_i(t,\tau)e_i(\tau).$$

Then, using (7.19), we obtain

$$\frac{\partial w}{\partial\tau}=\sum_{i=1}^m\left(\frac{\partial w_i}{\partial\tau}+\sum_{j=1}^m w_j\alpha_{j,i}\right)e_i+m^2w_mf_m,\tag{7.20}$$

$$\frac{\partial^2 w}{\partial t\partial\tau}=\sum_{i=1}^m\left(\frac{\partial^2 w_i}{\partial t\partial\tau}+\sum_{j=1}^m \frac{\partial w_j}{\partial t}\alpha_{j,i}\right)e_i+m^2\frac{\partial w_m}{\partial t}f_m.\tag{7.21}$$

Substituting (7.20) and (7.21) into (7.7) and comparing coefficients of e_i for $i=1,\dots,m$, we obtain the following system of equations:

$$\frac{\partial^2 w_i}{\partial t\partial\tau}-\frac{\frac{\partial w_m}{\partial t}}{w_m}\frac{\partial w_i}{\partial\tau}+m^2w_m^2w_i=\sum_{j=1}^m\left(\frac{\frac{\partial w_m}{\partial t}}{w_m}w_j-\frac{\partial w_j}{\partial t}\right)\alpha_{j,i},\quad 1\le i\le m.\tag{7.22}$$

The term in the right-hand side of (7.22), corresponding to $j = m$, is equal to zero. Hence Eq. (7.22) can be written in the form

$$\frac{\partial^2 w_i}{\partial t \partial \tau} - \frac{\frac{\partial w_m}{\partial t}}{w_m}\frac{\partial w_i}{\partial \tau} + m^2 w_m^2 w_i = \\ = \sum_{j=1}^{m-1}\left(\frac{\frac{\partial w_m}{\partial t}}{w_m} w_j - \frac{\partial w_j}{\partial t}\right)\alpha_{j,i},\ 1 \le i \le m. \tag{7.23}$$

By Remark 6, the system of Eqs. (7.23) does not depend on the choice of the subspace Δ.

In the same way Eq. (7.17) can be rewritten as an equation for components $Y_i(t,\tau) = \dfrac{w_i(t,\tau)}{w_m(t,\tau)}$ of the vector $Y_i(t,\tau)$ w.r.t. the canonical basis:

$$\frac{\partial^2 Y_i}{\partial t \partial \tau} + \left(\frac{\frac{\partial w_m}{\partial \tau}}{w_m}\right)\frac{\partial Y_i}{\partial t} + \left(\frac{\partial^2}{\partial t \partial \tau}(\ln w_m) + m^2 w_m^2\right) Y_i = \\ = -\sum_{j=1}^{m-1}\frac{\partial Y_j}{\partial t}\alpha_{j,i}. \tag{7.24}$$

4. Now we show that Eq. (7.23) (or (7.24)) has a "triangular" form. Note that by construction all functions $t \mapsto w(t,\tau)$ have singularities at $t = \tau$. Moreover, from item (1) of Corollary 4 it follows that their Laurent expansions at $t = \tau$ have the form

$$w_i(t,\tau) = \frac{1}{(t-\tau)^{m-i+1}} + \varphi_i(t,\tau), \tag{7.25}$$

where $\varphi_i(t,\tau)$ are smooth functions. Using this fact, one can obtain the following lemma.

Lemma 7.3. *The coefficients $\alpha_{j,i}(\tau)$, $1 \le j \le m-1$, satisfy the relations*

1. $\alpha_{j,i}(\tau) \equiv 0$ *if* $j < i-1$;
2. $\alpha_{i-1,i}(\tau) \equiv \dfrac{(i-1)(2m-i+1)}{m-i+1}$;
3. *if* $i \le j \le m-1$, *then* $\alpha_{j,i}(\tau)$ *can be expressed in terms of* $\dfrac{\partial^k}{\partial t^k}\varphi_m(t,\tau)\Big|_{t=\tau}$ *with* $0 \le k \le i-j$, *where* $\varphi_m(t,\tau)$ *is defined by* (7.25).

Proof. We will analyze the Laurent expansions of both sides of Eq. (7.23). We begin with the right-hand side. We denote

$$\Phi_m(t,\tau) = \frac{\partial}{\partial t}\ln\big(1 + (t-\tau)\varphi_m(t,\tau)\big). \tag{7.26}$$

Using (7.25), one can obtain the following series of relations:

$$\frac{\partial}{\partial t} w_j(t,\tau) = -\frac{m-j+1}{(t-\tau)^{m-j+2}} + O(1); \tag{7.27}$$

$$\frac{\frac{\partial}{\partial t} w_m}{w_m} = \frac{\partial}{\partial t}\ln w_m(t,\tau) = \frac{\partial}{\partial t}\ln\left(\frac{1}{t-\tau} + \varphi_m(t,\tau)\right) =$$
$$= -\frac{1}{t-\tau} + \Phi_m(t,\tau); \tag{7.28}$$

$$\frac{\frac{\partial}{\partial t} w_m}{w_m} w_j = -\frac{1}{(t-\tau)^{m-j+2}} + \frac{\Phi_m(t,\tau)}{(t-\tau)^{m-j+1}} + O\left(\frac{1}{t-\tau}\right). \tag{7.29}$$

Therefore, the right-hand side of (7.23) can be written in the form

$$\sum_{j=1}^{m-1}\left(\frac{m-j}{(t-\tau)^{m-j+2}} + \frac{\Phi_m(t,\tau)}{(t-\tau)^{m-j+1}}\right)\alpha_{j,i}(\tau) +$$
$$+ O\left(\frac{1}{t-\tau}\right). \tag{7.30}$$

Suppose that the function $t \mapsto \Phi_m(t,\tau)$ has the following expansion into the formal Taylor series at $t=\tau$:

$$\Phi_m(t,\tau) \approx \sum_{k=0}^{\infty} c_k(\tau)(t-\tau)^k. \tag{7.31}$$

Then by a direct computation we obtain that the right-hand side of (7.23) has the form

$$\frac{\alpha_{1,i}(\tau)}{(t-\tau)^{m+1}} + \sum_{j=2}^{m-1}\frac{(m-j)\alpha_{j,i}(\tau) + \sum_{k=1}^{j-1} c_{j-k-1}(\tau)\alpha_{k,i}(\tau)}{(t-\tau)^{m-j+2}} +$$
$$+ O\left(\frac{1}{(t-\tau)^2}\right). \tag{7.32}$$

Now we consider the left-hand side of (7.23). Using (7.26) and (7.25), we obtain the following series of relations:

$$\frac{\partial}{\partial \tau} w_i(t,\tau) = \frac{m-i+1}{(t-\tau)^{m-i+2}} + \frac{\partial}{\partial \tau}\varphi_i(t,\tau); \tag{7.33}$$

$$\frac{\partial^2}{\partial t\partial \tau} w_i(t,\tau) = -\frac{(m-i+1)(m-i+2)}{(t-\tau)^{m-i+3}} + O(1); \tag{7.34}$$

$$\frac{\frac{\partial}{\partial t} w_m}{w_m}\frac{\partial}{\partial \tau} w_i(t,\tau) = -\frac{m-i+1}{(t-\tau)^{m-i+3}} +$$

$$+\frac{\Phi_m(t,\tau)}{(t-\tau)^{m-i+2}}+O\left(\frac{1}{t-\tau}\right); \tag{7.35}$$

$$w_m(t,\tau)w_i(t,\tau)=\frac{1}{(t-\tau)^{m-i+3}}+\frac{2\varphi_m(t,\tau)}{(t-\tau)^{m-i+2}}+$$

$$+\frac{\varphi_m^2(t,\tau)}{(t-\tau)^{m-i+1}}+O\left(\frac{1}{(t-\tau)^2}\right). \tag{7.36}$$

Therefore, the left-hand side of (7.23) can be written in the form

$$\frac{m^2-(m-i+1)^2}{(t-\tau)^{m-i+3}}+\frac{1}{(t-\tau)^{m-i+2}}\Big(\Phi_m(t,\tau)+2\varphi_m(t,\tau)\Big)+$$

$$+\frac{\varphi_m^2(t,\tau)}{(t-\tau)^{m-i+1}}+O\biggl(\frac{1}{(t-\tau)^2}\biggr). \tag{7.37}$$

Comparing coefficients of (7.32) and (7.37), we have:

(1) if $m-j+2>m-i+3$, i.e., $j<i-1$, then $\alpha_{j,i}(\tau)\equiv 0$. This completes the proof of the first part of the lemma;
(2) if $m-j+2=m-i+3$, i.e., $j=i-1$, then $\alpha_{i-1,i}(\tau)(m-i+1)=m^2-(m-i+1)^2$. This completes the proof of the second part of the lemma;
(3) if $2<m-j+2<m-i+3$, i.e., $i-1<j<m$, then, taking into account that $\alpha_{k,i}(\tau)\equiv 0$ for $k<i-1$, we obtain

$$(m-j)\alpha_{j,i}(\tau)+\sum_{k=i-1}^{j-1}c_{j-k-1}(\tau)\alpha_{ki}(\tau)=c_{j-i}(\tau)+$$

$$+\frac{2\frac{\partial^{j-i}}{\partial t^{j-i}}\varphi_m(t,\tau)\Big|_{t=\tau}}{(j-i)!}+\frac{\frac{\partial^{j-i+1}}{\partial t^{j-i+1}}\varphi_m^2(t,\tau)\Big|_{t=\tau}}{(j-i+1)!}. \tag{7.38}$$

By (7.26), the coefficient $c_n(\tau)$ can be expressed in terms of $\frac{\partial^k}{\partial t^k}\varphi_m(t,\tau)\Big|_{t=\tau}$ with $0\le k\le n$. This together with (7.38) completes the proof of the third part of the lemma. □

By the previous lemma, Eq. (7.24) can be written for $2\le i\le m$ in the form

$$\frac{\partial Y_{i-1}}{\partial t}=-\frac{1}{\alpha_{i-1,i}}\left(\frac{\partial^2 Y_i}{\partial t\partial\tau}+\left(\frac{\frac{\partial w_m}{\partial\tau}}{w_m}\right)\frac{\partial Y_i}{\partial t}+\right.$$

$$\left.+\left(\frac{\partial^2}{\partial t\partial\tau}(\ln w_m)+m^2w_m^2\right)Y_i+\sum_{j=i}^{m-1}\frac{\partial Y_j}{\partial t}\alpha_{j,i}\right). \tag{7.39}$$

where $\alpha_{i-1,i} = \dfrac{(i-1)(2m-i+1)}{m-i+1}$. All terms in the right-hand side of (7.39) depend on the functions $Y_j(t,\tau)$ with $j \geq i$. Also we note that by (7.25)

$$Y_i(t,\tau) = \frac{w_i(t,\tau)}{w_m(t,\tau)} = \frac{1}{(t-\tau)^{m-i}} \frac{1+(t-\tau)^{m-i+1}\varphi_i(t,\tau)}{1+(t-\tau)\varphi_m(t,\tau)}. \qquad (7.40)$$

This implies that in the Laurent expansion of the function $t \mapsto Y(t,\tau)$ at $t=\tau$ all coefficients that correspond to nonpositive powers (and, in particular, the free term) depend on $w_m(t,\tau)$. This together with (7.39) yields that all $Y_i(t,\tau)$ (and, therefore, all $w_i(t,\tau)$) with $1 \leq i \leq m-1$ can be expressed in terms of $w_m(t,\tau)$. But by Remark 4 the components $w_i(t,\tau)$, $1 \leq i \leq m$, define the curve $\Lambda(t)$ uniquely, up to a symplectic transformation. This completes the proof of Theorem 1. $\square$

Now our goal is to find a complete system of symplectic invariants of the curve $\Lambda(t)$ of rank 1 and the constant weight, i.e., some set of functions of t which defines $\Lambda(t)$ uniquely, up to a symplectic transformation. By Theorem 1 it is natural to look for a complete system of invariants among coefficients $\beta_{i,j}(t)$ of expansion (4.16) of g into the Taylor series. Since $\Lambda(t)$ can be described, up to a symplectic transformation, by the curve $t \to w(t,\tau)$ of the vectors on the linear space of dimension m, it is natural to expect that a complete system of invariants of $\Lambda(\cdot)$ consists of m functions of t. By Lemma 4.1, the first m "independent" coefficients in expansion (4.16) are $\beta_{0,2i}(t)$ with $0 \leq i \leq m-1$. All these arguments lead to the following theorem.

Theorem 2. *The coefficients $\beta_{0,2i}(t)$, $0 \leq i \leq m-1$, define the curve $\Lambda(t)$ of rank 1 and a constant weight uniquely, up to a symplectic transformation.*

Let a function $\varphi_m(t,\tau)$ be as in (7.25). From identity (7.4) it easily follows that this theorem is equivalent to the following theorem.

Theorem 2′. *The functions $\tau \mapsto \dfrac{\partial^{2i-1}\varphi_m(t,\tau)}{\partial t^{2i-1}}\Big|_{t=\tau}$, $1 \leq i \leq m$, define the curve $\Lambda(t)$ of rank 1 and a constant weight uniquely, up to a symplectic transformation.*

Proof of Theorem 2′. Let functions $\varphi_i(t,\tau)$, $1 \leq i \leq m$, be as in (7.25). First, using the system of equations (7.23), we prove the following lemma.

Lemma 7.4. *All partial derivatives of the functions $\varphi_i(t,\tau)$, $1 \leq i \leq m$, at any diagonal point (τ,τ) can be expressed in terms of the functions $\tau \mapsto \dfrac{\partial^{2j-1}\varphi_m}{\partial t^{2j-1}}(t,\tau)\Big|_{t=\tau}$ and their derivatives, where $1 \leq j \leq m$.*

Proof. First, it is natural to make the change of coordinates $x = t - \tau$, $y = t + \tau$ such that the diagonal $t = \tau$ becomes the axis $x = 0$ in the new coordinates. Indeed, if we denote $z_i(x, y) = w_i\Bigl(\frac{x+y}{2}, \frac{y-x}{2}\Bigr)$, then system (7.23) can be transformed into the following system w.r.t. z_i:

$$-z_m\left(\frac{\partial^2 z_i}{\partial x^2} - \frac{\partial^2 z_i}{\partial y^2}\right) - \left(\frac{\partial z_m}{\partial x} + \frac{\partial z_m}{\partial y}\right)\left(\frac{\partial z_i}{\partial y} - \frac{\partial z_i}{\partial x}\right) + m^2 z_m^3 z_i =$$
$$= \sum_{j=i-1}^{m-1}\left(\left(\frac{\partial z_m}{\partial x} + \frac{\partial z_m}{\partial y}\right) z_j - \left(\frac{\partial z_j}{\partial x} + \frac{\partial z_j}{\partial y}\right) z_m\right)\alpha_{ji},\ 1 \le i \le m \tag{7.41}$$

(here we also have used the first part of Lemma 7.3). Relations (7.25) can be transformed into the relations

$$z_i(x, y) = \frac{1}{x^{m-i+1}} u_i(x, y), \tag{7.42}$$

where the functions $u_i(x, y)$ are smooth, $u_i(0, y) \equiv 1$, and $\dfrac{\partial^k}{\partial x^k} u_i(0, y) = 0$ for $1 \le i \le m$, $1 \le k \le m - i$. Substitute (7.42) into (7.41) and multiply both sides by x^{m-i+4}. Then we obtain some singular system of equations w.r.t. u_i. By a direct calculation it can be shown that this system has the following form:

$$x^2 u_m \frac{\partial^2 u_i}{\partial x^2} - (2m - 2i + 1) x u_m \frac{\partial u_i}{\partial x} + (m - i + 1) x u_i \frac{\partial u_m}{\partial x} +$$
$$+ \alpha_{i-1,i} x u_{i-1} \frac{\partial u_m}{\partial x} - \alpha_{i-1,i} x u_m \frac{\partial u_{i-1}}{\partial x} + (m - i + 1)^2 u_m u_i +$$
$$+ \alpha_{i-1,i}(m - i + 1) u_{i-1} u_m - m^2 u_m^3 u_i = \Psi_i, \tag{7.43}$$

where

$$\Psi_i = x^2 u_m \frac{\partial^2 u_i}{\partial y^2} + x u_m \frac{\partial u_i}{\partial y} - (m - i + 1) x u_i \frac{\partial u_m}{\partial y} -$$
$$- \alpha_{i-1,i} x u_{i-1} \frac{\partial u_m}{\partial y} +$$
$$+ \alpha_{i-1,i} x u_m \frac{\partial u_{i-1}}{\partial y} - x^2 \left(\frac{\partial u_m}{\partial x} + \frac{\partial u_m}{\partial y}\right)\left(\frac{\partial u_i}{\partial y} - \frac{\partial u_i}{\partial x}\right) -$$
$$- \sum_{j=i}^{m-1}\left(x^{j-i+2}\left(\frac{\partial u_m}{\partial x} + \frac{\partial u_m}{\partial y}\right) u_j - x^{j-i+2}\left(\frac{\partial u_j}{\partial x} + \frac{\partial u_j}{\partial y}\right) u_m +\right.$$
$$\left. + (m - j) x^{j-i+1} u_m u_j\right)\alpha_{ji}. \tag{7.44}$$

The left-hand side of Eq. (7.43) is a principal part of this equation in the following sense: differentiate both sides of (7.43) k times in x at the points of the initial curve $x = 0$. Then the right-hand side can be expressed in terms of the partial derivatives $\dfrac{\partial^n u_p}{\partial x^n}(0, y)$ with n *less than* k and their derivatives w.r.t. y (here one can take $i - 1 \leq p \leq m$), while any term of the left-hand side (at least for $k \geq 2$) depends also on a partial derivative of some u_j w.r.t. x of *order* k at $(0, y)$. Moreover, using the fact that $u_i(0, y) \equiv 1$, $1 \leq i \leq m$, and $\alpha_{i-1,i} = \dfrac{(2m - i + 1)(i - 1)}{m - i + 1}$, one can easily obtain in this way the following linear system w.r.t. $\dfrac{\partial^k}{\partial x^k} u_i(0, y)$, $1 \leq i \leq m$, for a given integer $k \geq 0$:

$$\zeta_i(k)\frac{\partial^k u_{i-1}}{\partial x^k}(0, y) + \eta_i(k)\frac{\partial^k u_i}{\partial x^k}(0, y) + \theta_i(k)\frac{\partial^k u_m}{\partial x^k}(0, y) = \widetilde{\Psi}_i, \quad 1 \leq i \leq m, \tag{7.45}$$

where

$$\begin{aligned} \zeta_i(k) &= \frac{(k + i - m - 1)(i - 2m - 1)(i - 1)}{m - i + 1}, \\ \eta_i(k) &= (k + i - 1)(k + i - 2m - 1), \\ \theta_i(k) &= \frac{k + 2i - 2 - 2m}{m - i + 1} m^2, \end{aligned} \tag{7.46}$$

and $\widetilde{\Psi}_i$ can be expressed in terms of the partial derivatives of the form $\dfrac{\partial^n u_p}{\partial x^n}(0, y)$ with n *less than* k and their derivatives w.r.t. y (here $i - 1 \leq p \leq m$).

It turns out that the determinant of system (7.45) satisfies the following remarkable identity:

$$\begin{vmatrix} \eta_1(k) & 0 & 0 & \dots & 0 & 0 & \theta_1(k) \\ \zeta_2(k) & \eta_2(k) & 0 & \dots & 0 & 0 & \theta_2(k) \\ 0 & \zeta_3(k) & \eta_3(k) & \dots & 0 & 0 & \theta_3(k) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \zeta_{m-1}(k) & \eta_{m-1}(k) & \theta_{m-1}(k) \\ 0 & 0 & 0 & \dots & 0 & \zeta_m(k) & \eta_m(k) + \theta_m(k) \end{vmatrix} = \\ = \prod_{j=1}^{m} (k - 2j)(k + 2j - 1). \tag{7.47}$$

The proof of (7.47) that we have found is rather long and will be presented in Appendix.

As a consequence of (7.47), we obtain that the determinant of system (7.45) has exactly m positive zeros at $k = 2j$, $1 \le j \le m$. Therefore, any partial derivative of u_i, $1 \le i \le m$, at $(0, y)$ can be expressed in terms of the functions $y \mapsto \dfrac{\partial^{2j} u_p}{\partial x^{2j}}(0, y)$ and their derivatives, where $1 \le j, p \le m$. Moreover, by Theorem 1 $u_p(x, y)$ can be expressed in terms of $u_m(x, y)$ and its derivatives. Hence any partial derivative of u_i, $1 \le i \le m$, at $(0, y)$ can be expressed in terms of the functions $y \mapsto \dfrac{\partial^{2j} u_m}{\partial x^{2j}}(0, y)$ and their derivatives, where $1 \le j \le m$. But this is equivalent to the statement of our lemma, if we return to the old coordinates t and τ. $\square$

Now we define a canonical moving frame: for given τ take the derivative subspace $\Lambda^0(\tau)$ and let $f_1(\tau), \dots, f_m(\tau)$ be a basis of $\Lambda^0(\tau)$ dual to the canonical basis of $\Lambda(\tau)$ (i.e., $\sigma(f_i(\tau), e_j(\tau)) = \delta_{i,j}$). The basis

$$(e_1(\tau), \dots, e_m(\tau), f_1(\tau), \dots, f_m(\tau))$$

of the whole symplectic space W is called *the canonical moving frame* of the curve $\Lambda(\cdot)$. Denote by $E(\tau)$ and $F(\tau)$ the tuples of vectors $(e_1(\tau), \dots, e_m(\tau))$ and $(f_1(\tau), \dots, f_m(\tau))$, respectively, arranged in the columns. Denote by S_t the matrix, corresponding to the linear mapping $\langle \Lambda(\tau), \Lambda(t), \Lambda^0(\tau) \rangle$ w.r.t. to the canonical basis, and by S_t^0 the matrix, corresponding to the linear mapping $\langle \Lambda^0(\tau), \Lambda^0(t), \Lambda(\tau) \rangle$ w.r.t. to the basis $(f_1(\tau), \dots, f_m(\tau))$ (see Sec. 2 for the notation). Also, let $\Omega(\tau)$ be an $m \times m$ matrix with (i, j) entry equal to $\alpha_{i,j}(\tau)$, where $\alpha_{i,j}(\tau)$ is defined by (7.19) with $\Delta = \Lambda^0(\tau)$. Then it is easy to see that the structural equation for the canonical moving frame has the following form:

$$\begin{pmatrix} \dot{E}(\tau) \\ \dot{F}(\tau) \end{pmatrix} = \begin{pmatrix} \Omega(\tau) & \dot{S}_\tau \\ \dot{S}_\tau^0 & -\Omega^T(\tau) \end{pmatrix} \begin{pmatrix} E(\tau) \\ F(\tau) \end{pmatrix}. \tag{7.48}$$

We claim that in order to prove Theorem 2′, it is sufficient to prove the following lemma.

Lemma 7.5. *The matrix in the structural equation* (7.48) *depends only on the coefficients of the expansions of* $t \to w_i(t, \tau)$*,* $1 \le i \le m$*, in the Laurent series at* $t = \tau$*.*

Indeed, if Lemma 7.5 holds, then first by Lemma 7.4 this matrix depends only on the functions $\tau \mapsto \dfrac{\partial^{2j-1} \varphi_m}{\partial t^{2j-1}}(t, \tau)\Big|_{t=\tau}$, $1 \le j \le m$, second, the structural equation (7.48) has a unique solution with a prescribed initial condition, and, finally, any symplectic basis can be taken as an initial condition of (7.48).

Proof of Lemma 7.5. First, according to (7.3),

$$(\dot{S}_\tau))_{i,j} = \begin{cases} 0, & (i,j) \neq (m,m), \\ m^2, & (i,j) = (m,m). \end{cases} \tag{7.49}$$

Further, by recursive formula (2.4) for $i = 0$:

$$\begin{aligned} \dot{S}^0_\tau = \frac{d}{d\tau} A_0(\tau) = A_1(\tau) + \\ + \sum_{n=1-2m}^{-1} \Big(A_n(\tau) \dot{S}_\tau A_{-n}(\tau) + A_{-n}(\tau) \dot{S}_\tau A_n(\tau) \Big), \end{aligned} \tag{7.50}$$

where $A_j(\tau)$ are defined by expansion (2.3) (here we have used the fact that by definition of the derivative curve $A_0(\tau) = 0$ and by Lemma 6.1 the order of the pole of $t \mapsto (S_t - S_\tau)^{-1}$ at $t = \tau$ is equal to $2m-1$). By the definition of the vectors $w(t,\tau)$, we have

$$\Big((S_t - S_\tau)^{-1} \Big)_{i,j} = - \int^t w_i(\xi,\tau) w_j(\xi,\tau) d\xi.$$

Therefore, any $A_n(\tau)$ with $n \neq 0$ can be expressed in terms of the coefficients of the expansions of $t \to w_i(t,\tau)$, $1 \leq i \leq m$, into the Laurent series at $t = \tau$. This together with (7.49) and (7.50) implies that $\dot{S}^0_\tau$ can be expressed in terms of the coefficients of the expansions of $t \to w_i(t,\tau)$, $1 \leq i \leq m$, into the Laurent series at $t = \tau$.

Finally, let us analyze the matrix $\Omega(\tau)$. By Lemma 7.3, all its entries $\alpha_{i,j}(\tau)$ with $1 \leq i \leq m-1$ can be expressed in terms of the coefficients of the expansions of $t \to w_m(t,\tau)$ into the Laurent series at $t = \tau$. The entries $\alpha_{m,j}(\tau)$ do not enter the differential equation (7.23). To find an expression for these entries, we will use the integral-differential equation (7.12) that can be rewritten for $\Delta = \Lambda_0(\tau)$ in the form

$$\frac{\partial w^{\Lambda_0(\tau)}(t,\tau_1)}{\partial \tau_1}\Big|_{\tau_1=\tau} = -m^2 w_m(t,\tau) \int^t w_m(\xi,\tau) w(\xi,\tau) d\xi. \tag{7.51}$$

Using (7.20), we can obtain from here the following system of equation w.r.t. the components $w_j(t,\tau)$:

$$\begin{aligned} \frac{\partial w_j(t,\tau)}{\partial \tau} + \sum_{l=1}^m w_l(t,\tau) \alpha_{lj}(\tau) = \\ = -m^2 w_m(t,\tau) \int^t w_m(\xi,\tau) w_j(\xi,\tau) d\xi, \quad 1 \leq j \leq m. \end{aligned} \tag{7.52}$$

For a given j consider the Laurent expansion of the left-hand side of (7.51), as a function of t, at $t=\tau$. By (7.25), the coefficient of $\dfrac{1}{t-\tau}$ in this expansion is equal to $\alpha_{m,j}(\tau)$. On the other hand, all coefficients of the appropriate expansion of the right-hand side can be expressed in terms of coefficients of expansions of $t \to w_j(t,\tau)$ and $t \to w_m(t,\tau)$ into the Laurent series at $t=\tau$. Therefore, the entries $\alpha_{m,j}(\tau)$ can also be expressed in terms of coefficients of expansions of $t \to w_i(t,\tau)$ (even with $i=j$ or m) into the Laurent series at $t=\tau$. This concludes the proof of our lemma and also of Theorem $2'$. $\square$

8. Appendix

In this appendix we prove identity (7.47). We are sure that the proof presented here is far from being optimal, but this is the only one that we have at this moment.

Denote the determinant in the left-hand side of (7.47) by $L_m(k)$. Expanding this determinant w.r.t. the last column, we have

$$L_m(k)=\sum_{j=1}^{m}(-1)^{j+m}\theta_j(k)\prod_{i=1}^{j-1}\eta_i(k)\prod_{i=j+1}^{m}\zeta_i(k)+\prod_{i=1}^{m}\eta_i(k). \tag{8.1}$$

Then, substituting (7.46) into (8.1), one can easily transform $L_m(k)$ to the following form:

$$\begin{aligned}L_m(k)=\sum_{j=1}^{m+1}&\frac{m(2m-j)}{(m-j+1)!(j-1)}\big(k-2(m-j+1)\big)\times\\&\times\prod_{i=2-j}^{m-j}(k-i)\prod_{i=2m-j+2}^{2m}(k-i).\end{aligned} \tag{8.2}$$

Note that $L_m(k)$ is a polynomial of degree $2m$, exactly as the polynomial in the right-hand side of (7.47). Also, for both polynomials the coefficient of leading term k^{2m} is equal to 1. Therefore, in order to prove identity (7.47), it is sufficient to prove that the polynomials in both sides of (7.47) have the same roots, or, equivalently, that $L_m(2i)=L_m(1-2i)=0$ for all $1\le i\le m$.

We will do this in two steps: first we will show that

$$L_m(2i)=0,\quad 1\le i\le m. \tag{8.3}$$

Second, we will prove that the function

$$\overline{L_m}(k)\overset{\text{def}}{=}\frac{k}{k-2m}L_m(k) \tag{8.4}$$

satisfies

$$\overline{L_m}(-1-k) = \overline{L_m}(k), \tag{8.5}$$

i.e., $\overline{L_m}(k)$ is invariant under the reflection of its argument w.r.t. $-1/2$. This together with (8.3) and the fact that $\overline{L_m}(0) = 0$ (which follows directly from the definition of $\overline{L_m}(k)$) will imply that also $L_m(1-2i) = 0$ for all $1 \le i \le m$.

1. The proof of (8.3). For $1 \le j \le m+1$, we denote

$$p_{m,j}(k) = \frac{m(2m-j)!}{(m-j+1)!(j-1)!}\Big(k - 2(m-j+1)\Big) \times \\ \times \prod_{i=2-j}^{m-j}(k-i) \prod_{i=2m-j+2}^{2m}(k-i). \tag{8.6}$$

By a direct computation, the following identity can be easily verified:

$$p_{m,j}(2m-2l) + p_{m,2l+2-j}(2m-2l) = 0, \tag{8.7}$$

where

$$0 \le l \le m-1, \quad \max\{1, 2l+1-m\} \le j \le \min\{m+1, 2l+1\}.$$

In particular, applying (8.6) to $j = l+1$, we have

$$p_{m,l+1}(2m-2l) = 0. \tag{8.8}$$

By construction,

$$L_m(2m-2l) = \sum_{j=1}^{m+1} p_{m,j}(2m-2l). \tag{8.9}$$

Denote $l_1 = \max\{1, 2l+1-m\}$ and $l_2 = \min\{m+1, 2l+1\}$. Consider the following 3 cases:

(1) $l_1 \le j \le l_2$. Then from (8.7) and (8.8) it follows that

$$\sum_{j=l_1}^{l_2} p_{m,j}(2m-2l) = \sum_{j=l_1}^{l}\Big(p_{m,j}(2m-2l) + p_{m,2l+2-j}(2m-2l)\Big) + \\ + p_{m,l+1}(2m-2l) = 0; \tag{8.10}$$

(2) $2l+2 \le j \le m+1$. Then $2m-j+2 \le m-l \le 2m$; therefore, from (8.6) it follows that $p_{m,j}(2m-2l) = 0$;

(3) $1 \le j \le 2l-m$. Then $2 \le 2m-2l \le m-j$ and again from (8.6) it follows that in this case $p_{m,j}(2m-2l) = 0$.

Therefore, by (8.9), $L_m(2m-2l) = 0$ for all $0 \le l \le m-1$, or, equivalently, $L_m(2i) = 0$ for all $1 \le i \le m$.

2. The proof of (8.5). We will transform the expression for $L_m(k)$ into a more symmetric form. Following [5] (Chap. 1, Sec. 2) we denote

$$x^{n|h} \stackrel{\text{def}}{=} x(x+h)\dots(x+(n-1)h). \tag{8.11}$$

Then similarly to the Newton binomial identity, one easily obtains

$$(x+y)^{n|h} = \sum_{i=0}^{n} \binom{n}{i} x^{n-i|h} y^{i|h}. \tag{8.12}$$

Using notation (8.11), one can rewrite $L_m(k)$ in the form

$$\begin{aligned} L_m(k) = \sum_{j=1}^{m+1} \frac{m(2m-j)!}{(m-j+1)!(j-1)!}(k-2(m-j+1)) \times \\ \times (k-m+j)^{m-1|1}(k-2m)^{j-1|1}. \end{aligned} \tag{8.13}$$

Applying (8.12), one obtains

$$\begin{aligned} (k-m+j)^{m-1|1} &= \big((k+1)+(j-m-1)\big)^{m-1|1} = \\ &= \sum_{i=0}^{m-1} \binom{m}{i} (k+1)^{m-1-i|1}(j-m-1)^{i|1} = \\ &= \sum_{i=0}^{m-j+1} (-1)^i \binom{m}{i} (k+1)^{m-1-i|1} \frac{(m-j+1)!}{(m-j-i+1)!}. \end{aligned} \tag{8.14}$$

Substituting (8.14) into (8.13) and changing the order of summation, one easily obtains

$$\begin{aligned} L_m(k) = \sum_{i=0}^{m} \left(\sum_{j=0}^{m-i} \frac{(2m-j-1)!}{j!(m-j-i)!}(k-2(m-j))(k-2m)^{j|1} \right) \times \\ \times \frac{(-1)^i m!(k+1)^{m-1-i|1}}{(m-i-1)!i!}. \end{aligned} \tag{8.15}$$

Lemma 8.1. *The following identity holds*:

$$\begin{aligned} \sum_{j=0}^{m-i} \frac{(2m-j-1)!}{j!(m-j-i)!}(k-2m)^{j|1}(k-2(m-j)) = \\ = \frac{(m+i-1)!}{(m-i)!}(k-2m)(k+m-i) \prod_{l=1}^{m-i-1} (k-l). \end{aligned} \tag{8.16}$$

Proof. Using the representation $k-2(m-j)=(k-2m)+2j$, one can split the left-hand side of (8.16) into the sum of two terms:

$$(k-2m)\sum_{j=0}^{m-i}\frac{(2m-j-1)!}{j!(m-j-i)!}(k-2m)^{j|1}+$$
$$+2\sum_{j=1}^{m-i}\frac{(2m-j-1)!}{(j-1)!(m-j-i)!}(k-2m)^{j|1}. \tag{8.17}$$

Since

$$(2m-j-1)=(m+i-1)!(m+i)^{m-i-j|1},$$

the first term of (8.17) can be written in such a way that one can apply the binomial identity (8.12):

$$\frac{(m+i-1)!(k-2m)}{(m-i)!}\sum_{j=0}^{m-i}\binom{m-i}{j}(m+i)^{m-i-j|1}(k-2m)^{j|1}=$$
$$=\frac{(m+i-1)!(k-2m)}{(m-i)!}(k-m+i)^{m-i|1}. \tag{8.18}$$

In the same way, the second term of (8.17) can also be written in such a way that one can apply the binomial identity (8.12):

$$2\sum_{j=1}^{m-i}\frac{(2m-j-1)!}{(j-1)!(m-j-i)!}(k-2m)^{j|1}=$$
$$=2\sum_{j=0}^{m-i-1}\frac{(2m-j-2)!}{(j)!(m-j-i-1)!}(k-2m)^{j+1|1}=$$
$$=2(k-2m)\sum_{j=0}^{m-i-1}\frac{(2m-j-2)!}{(j)!(m-j-i-1)!}(k-2m+1)^{j|1}=$$
$$=\frac{2(m+i-1)!(k-2m)}{(m-i-1)!}\times$$
$$\times\sum_{j=1}^{m-i-1}\binom{m-i-1}{j}(m+i)^{m-i-1-j|1}(k-2m+1)^{j|1}=$$
$$=\frac{2(m+i-1)!(k-2m)}{(m-i-1)!}(k-m+i+1)^{m-i-1|1}. \tag{8.19}$$

Combining (8.18) and (8.19), we obtain that the left-hand side of (8.16) is equal to

$$\frac{(m+i-1)!(k-2m)}{(m-i-1)!}\Big(\frac{k-m+i}{m-i}+2\Big)(k-m+i+1)^{m-i-1|1}=$$

$$= \frac{(m+i-1)!}{(m-i)!}(k-2m)(k+m-i)(k-m+i+1)^{m-i-1|1} =$$

$$= \frac{(m+i-1)!}{(m-i)!}(k-2m)(k+m-i)\prod_{l=1}^{m-i-1}(k-l), \tag{8.20}$$

which is exactly the right-hand side of (8.16). This completes the proof of the lemma. □

Now substituting (8.16) into (8.15), we have the following identity:

$$L_m(k) = (k-2m)\sum_{i=0}^{m}\frac{(-1)^i m!(m+i-1)!}{i!(m-i)!(m-i-1)!}\times$$
$$\times \prod_{l=1}^{m-i-1}(k-l)\prod_{l=1}^{m-i}(k+l). \tag{8.21}$$

Then the function $\overline{L_m}(k)$ satisfies

$$\overline{L_m}(k) = \frac{k}{k-2m}L_m(k) =$$
$$= \sum_{i=0}^{m}\frac{(-1)^i m!(m+i-1)!}{i!(m-i)!(m-i-1)!}\prod_{l=1}^{m-i}(k+1-l)\prod_{l=1}^{m-i}(k+l). \tag{8.22}$$

It remains only to note that all terms of the sum in the right-hand side of (8.22) are invariant under the reflection of the argument w.r.t. $-\frac{1}{2}$ or, equivalently, under the substitution $k \to -1-k$. Then the function $\overline{L_m}(k)$ is also invariant under this substitution, which proves (8.5) and, therefore, also (7.47).

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