Principal Invariants of Jacobi Curves

Andrei Agrachev[1] and Igor Zelenko[2]

[1] S.I.S.S.A., Via Beirut 2-4, 34013 Trieste, Italy and Steklov Mathematical Institute, ul. Gubkina 8, 117966 Moscow, Russia; email: agrachev@sissa.it

[2] Department of Mathematics, Technion-Israel Institute of Technology, Haifa 32000, Israel; email: zigor@techunix.technion.ac.il

Abstract. Jacobi curves are far going generalizations of the spaces of "Jacobi fields" along Riemannian geodesics. Actually, Jacobi curves are curves in the Lagrange Grassmannians. Differential geometry of these curves provides basic feedback or gauge invariants for a wide class of smooth control systems and geometric structures. In the present paper we mainly discuss two principal invariants: the *generalized Ricci curvature*, which is an invariant of the parametrized curve in the Lagrange Grassmannian providing the curve with a natural projective structure, and a *fundamental form*, which is a 4-order differential on the curve. This paper is a continuation of the works [1, 2], where Jacobi curves were defined, although it can be read independently.

1 Introduction

Suppose M is a smooth manifold and $\pi : T^*M \to M$ is the cotangent bundle to M. Let H be a codimension 1 submanifold in T^*M such that H is transversal to T_q^*M, $\forall q \in M$; then $H_q = H \cap T_qM$ is a smooth hypersurface in T_qM. Let ς be the canonical Liouville form on T_q^*M, $\varsigma_\lambda = \lambda \circ \pi_*$, $\lambda \in T^*M$, and $\sigma = d\varsigma$ be the standard symplectic structure on T^*M; then $\sigma|_H$ is a corank 1 closed 2-form. The kernels of $(\sigma|_H)_\lambda$, $\lambda \in H$ are transversal to T_q^*M, $q \in M$; these kernels form a line distribution in H and define a *characteristic 1-foliation* $\mathcal{C}$ of H. Leaves of this foliation are *characteristic curves* of $\sigma|_H$.

Suppose γ is a segment of a characteristic curve and O_γ is a neighborhood of γ such that $N = O_\gamma/(\mathcal{C}|_{O_\gamma})$ is a well-defined smooth manifold. The quotient manifold N is in fact a symplectic manifold endowed with a symplectic structure $\bar\sigma$ induced by $\sigma|_H$. Let $\phi : O_\gamma \to N$ be the canonical factorization; then $\phi(H_q \cap O_\gamma)$, $q \in M$, are Lagrangian submanifolds in N. Let $L(T_\gamma N)$ be the Lagrange Grassmannian of the symplectic space $T_\gamma N$, i.e. $L(T_\gamma N) = \{\Lambda \subset T_\gamma N : \Lambda^\angle = \Lambda\}$, where $D^\angle = \{e \in T_\gamma N : \bar\sigma(e, D) = 0\}$, $\forall D \subset T_\gamma N$. *Jacobi curve* is the mapping

$$\lambda \mapsto \phi_*(T_\lambda H_{\pi(\lambda)}), \quad \lambda \in \gamma,$$

from γ to $L(T_\gamma N)$.

Jacobi curves are curves in the Lagrange Grassmannians. They are invariants of the hypersurface H in the cotangent bundle. In particular, any differential invariant of the curves in the Lagrange Grassmannian by the action of the linear Symplectic Group produces a well-defined function on H.

Set $W = T_\gamma N$ and note that the tangent space $T_\Lambda L(W)$ to the Lagrange Grassmannian at the point Λ can be naturally identified with the space of quadratic forms on the linear space $\Lambda \subset W$. Namely, take a curve $\Lambda(t) \in L(W)$ with $\Lambda(0) = \Lambda$. Given some vector $l \in \Lambda$, take a curve $l(\cdot)$ in W such that $l(t) \in \Lambda(t)$ for all t and $l(0) = l$. Define the quadratic form $q_{\Lambda(\cdot)}(l) = \bar{\sigma}(\frac{d}{dt}l(0), l)$. Using the fact that the spaces $\Lambda(t)$ are Lagrangian, i.e. $\Lambda(t)^\angle = \Lambda(t)$, it is easy to see that the form $q_{\Lambda(\cdot)}(l)$ depends only on $\frac{d}{dt}\Lambda(0)$. So, we have the map from $T_\Lambda L(W)$ to the space of quadratic forms on Λ. A simple counting of dimension shows that this mapping is a bijection.

Proposition 1. *Tangent vectors to the Jacobi curve J_γ at a point $J_\gamma(\lambda)$, $\lambda \in \gamma$, are equivalent (under linear substitutions of variables in the correspondent quadratic forms) to the "second fundamental form" of the hypersurface $H_{\pi(\lambda)} \subset T^*_{\pi(\lambda)}M$ at the point λ.*

In particular, the velocity of J_γ at λ is a sign-definite quadratic form if and only if the hypersurface $H_{\pi(\lambda)}$ is strongly convex at λ.

A similar construction can be done for a submanifold of codimension 2 in T^*M. In the codimension 2 case characteristic curves do not fill the whole submanifold; they are concentrated in the characteristic variety consisting of the points, where the restriction of σ to the submanifold is degenerate.

We are mainly interested in submanifolds that are dual objects to smooth control systems. Here we call a smooth control system any submanifold $V \subset TM$, transversal to fibers. Let $V_q = V \cap T_qM$; The "dual" *normal variety* H^1 and *abnormal variety* H^0 are defined as follows:

$$H^1 = \bigcup_{q \in M} \{\lambda \in T^*_q M : \exists v \in V_q,\ \langle \lambda, v\rangle = 1,\ \langle \lambda, T_v V_q\rangle = 0\},$$

$$H^0 = \bigcup_{q \in M} \{\lambda \in T^*_q M \setminus 0 : \exists v \in V_q,\ \langle \lambda, v\rangle = \langle \lambda, T_v V_q\rangle = 0\}.$$

These varieties are not, in general, smooth manifolds; they may have singularities, which we do not discuss here. Anyway, one can obtain a lot of information on the original system just studying smooth parts of H^1, H^0.

Characteristic curves of $\sigma|^1_H$ $(\sigma|^0_H)$ are associated with normal (abnormal) extremals of the control system V. The corresponding Jacobi curves admit a purely variational construction in terms of the original control system and in a very general setting (singularities included), see [1, 2, 3].

One of the varieties H^1, H^0 can be empty. In particular, if $V_q = \partial W_q$, where W_q is a convex set and $0 \in intW_q$, then $H^0 = \emptyset$. Moreover, in this case the Liouville form never vanishes on the tangent lines to the characteristic curves of $\sigma|_{H^1}$, and any characteristic curve γ has a canonical parametrization by the rule $\langle \varsigma, \dot{\gamma}\rangle = 1$.

If subsets $V_q \subset T_qM$ are conical, $\alpha V_q = V_q$, $\forall \alpha > 0$, then, in contrast to the previous case, $H^1 = \emptyset$ and ς vanishes on the tangent lines to the characteristic curves of $\sigma|_{H^0}$. The characteristic curves are actually unparametrized.

If V_q are compact, then H^1 has codimension 1 in T^*M, while H^0 has codimension ≥ 2 in all nontrivial cases.

The rank of the "second fundamental form" of the submanifolds H_q^1 and H_q^0 of T_q^*M at any point is no greater than $\dim V_q$. Indeed, let $\lambda \in H_q^1$; then $\lambda \in (T_vV_q)^\perp$, $\langle \lambda, v \rangle = 1$, for some $v \in V_q$. We have $\lambda + (T_vV_q + \mathbb{R}v)^\perp \subset H_q^1$. So λ belongs to an affine subspace of dimension $n - \dim V_q - 1$, which is contained in H_q^1. For $\lambda \in H_q^0$, $\exists\, v \in V_q$ such that $\lambda \in (T_vV_q)^\perp$, $\langle \lambda, v \rangle = 0$. Then the affine subspace $\lambda + (T_vV_q + \mathbb{R}v)^\perp$ is contained in H_q^0.

Suppose that H^1 has codimension 1 in T^*M and γ is a characteristic curve of $\sigma|_{H^1}$. Then the velocity of the Jacobi curve $\lambda \mapsto J_\gamma(\lambda)$, $\lambda \in \gamma$, has rank no greater than $\dim V_{\pi(\lambda)}$ (see proposition 1). The same is true for the Jacobi curves associated with characteristic curves of $\sigma|_{H^0}$, if H^0 has codimension 2.

Dimension of V_q is the number of inputs or control parameters in the control system. Less inputs means more "nonholonomic constraints" on the system. It happens that the rank of velocity of any Jacobi curve generated by the system never exceeds the number of inputs.

2 Derivative Curve

Let Λ be a Lagrangian subspace of W, i.e. $\Lambda \in L(W)$. For any $w \in \Lambda$, the linear form $\bar\sigma(\cdot, w)$ vanishes on Λ and thus defines a linear form on W/Λ. The nondegeneracy of $\bar\sigma$ implies that the relation $w \mapsto \sigma(\cdot, w)$, $w \in \Lambda$, induces a canonical isomorphism $\Lambda \cong (W/\Lambda)^*$ and, by the conjugation, $\Lambda^* \cong W/\Lambda$.

We set $\Lambda^\pitchfork = \{\Gamma \in L(W) : \Gamma \cap \Lambda = 0\}$, an open everywhere dense subset of $L(W)$. Let $Sym^2(\Lambda)$ be the space of self-adjoint linear mappings from Λ^* to Λ; this notation reflects the fact that $Sym^2(\Lambda)$ is the space of quadratic forms on Λ^* that is the symmetric square of Λ. $\Lambda^\pitchfork$ possesses a canonical structure of an affine space over the linear space $Sym^2(\Lambda) = Sym^2((W/\Lambda)^*)$. Indeed, for any $\Delta \in \Lambda^\pitchfork$ and coset $(w+\Lambda) \in W/\Lambda$, the intersection $\Delta \cap (w+\Lambda)$ of the linear subspace Δ and the affine subspace $w+\Lambda$ in W consists of exactly one point. To a pair $\Gamma, \Delta \in \Lambda^\pitchfork$ there corresponds a mapping $(\Gamma - \Delta) : W/\Lambda \to \Lambda$, where

$$(\Gamma - \Delta)(w + \Lambda) \stackrel{def}{=} \Gamma \cap (w + \Lambda) - \Delta \cap (w + \Lambda).$$

It is easy to check that the identification $W/\Lambda = \Lambda^*$ makes $(\Gamma - \Delta)$ a self-adjoint mapping from Λ^* to Λ. Moreover, given $\Delta \in \Lambda^\pitchfork$, the correspondence $\Gamma \mapsto (\Gamma - \Delta)$ is a one-to-one mapping of $\Lambda^\pitchfork$ onto $Sym^2(\Lambda)$ and the axioms of the affine space are obviously satisfied.

Fixing $\Delta \in \Lambda^\pitchfork$ one obtains a canonical identification $\Delta \cong W/\Lambda = \Lambda^*$. In particular, $(\Gamma - \Delta) \in Sym^2(\Lambda)$ turns into the mapping from Δ to Λ. For the last linear mapping we will use the notation $\langle \Delta, \Gamma, \Lambda\rangle : \Delta \to \Lambda$. In fact, this mapping has a much more straightforward description. Namely, the relations $W = \Delta \oplus \Lambda$, $\Gamma \cap \Lambda = 0$, imply that Γ is the graph of a linear mapping from Δ to Λ. Actually, it is the graph of the mapping $\langle \Delta, \Gamma, \Lambda\rangle$. In particular, $\ker\langle \Delta, \Gamma, \Lambda\rangle = \Delta \cap \Gamma$. If $\Delta \cap \Gamma = 0$, then $\langle \Lambda, \Gamma, \Delta\rangle = \langle \Delta, \Gamma, \Lambda\rangle^{-1}$.

Let us give coordinate representations of the introduced objects. We may assume that

$$W = \mathbb{R}^m \oplus \mathbb{R}^m = \{(x, y) : x, y \in \mathbb{R}^m\},$$

$$\bar\sigma((x_1, y_1), (x_2, y_2)) = \langle x_1, y_2\rangle - \langle x_2, y_1\rangle,\ \Lambda = \mathbb{R}^m \oplus 0,\ \Delta = 0 \oplus \mathbb{R}^m.$$

Then any $\Gamma \in \Delta^{\pitchfork}$ takes the form $\Gamma = \{(x, Sx) : x \in \mathbb{R}^n\}$, where S is a symmetric $m \times m$ matrix. The operator $\langle \Lambda, \Gamma, \Delta\rangle : \Lambda \to \Delta$ is represented by the matrix S, while the operator $\langle \Delta, \Gamma, \Lambda\rangle$ is represented by the matrix S^{-1}.

The coordinates in Λ induce the identification of $Sym^2\Lambda$ with the space of symmetric $m\times m$ matrices. $\Lambda^{\pitchfork}$ is an affine subspace over $Sym^2\Lambda$; we fix Δ as the origin in this affine subspace and thus obtain a coordinatization of $\Lambda^{\pitchfork}$ by symmetric $m\times m$ matrices. In particular, the "point" $\Gamma = \{(x, Sx) : x \in \mathbb{R}^n\}$ in $\Lambda^{\pitchfork}$ is represented by the matrix S^{-1}.

A subspace $\Gamma_0 = \{(x, S_0x) : x \in \mathbb{R}^n\}$ is transversal to Γ if and only if $det(S - S_0) \neq 0$. Let us pick coordinates $\{x\}$ in Γ_0 and fix Δ as the origin in the affine space $\Gamma_0^{\pitchfork}$. In the induced coordinatization of $\Gamma_0^{\pitchfork}$ the "point" Γ is represented by the matrix $(S - S_0)^{-1}$.

Let $t \mapsto \Lambda(t)$ be a smooth curve in $L(W)$. We say that the curve $\Lambda(\cdot)$ is *ample at* τ if $\exists k > 0$ such that for any representative $\Lambda^k_\tau(\cdot)$ of the k-jet of $\Lambda(\cdot)$ at τ, $\exists t$ such that $\Lambda^k_\tau(t) \cap \Lambda(\tau) = 0$. The curve $\Lambda(\cdot)$ is called *ample* if it is ample at any point.

We have given an intrinsic definition of an ample curve. In coordinates it takes the following form: the curve $t \mapsto \{(x, S_tx) : x \in \mathbb{R}^n\}$ is ample at τ if and only if the function $t \mapsto det(S_t - S_\tau)$ has a root of *finite order* at τ.

Assume that $\Lambda(\cdot)$ is ample at τ. Then $\Lambda(t) \in \Lambda(\tau)^{\pitchfork}$ for all t from a punctured neighborhood of τ. We obtain the curve $t \mapsto \Lambda(t) \in \Lambda(\tau)^{\pitchfork}$ in the affine space $\Lambda(\tau)^{\pitchfork}$ with the pole at τ. Fixing an "origin" in $\Lambda(\tau)^{\pitchfork}$ we make $\Lambda(\cdot)$ a vector function with values in $Sym^2(\Lambda)$ and with the pole at τ. Such a vector function admits the expansion in the Laurent series at τ. Obviously, only free term in the Laurent expansion depends on the choice of the "origin" we did to identify the affine space with the linear one. More precisely, the addition of a vector to the "origin" results in the addition of the same vector to the free term in the Laurent expansion. In other words, for the Laurent expansion of a curve in an affine space, the free term of the expansion is a point of this affine space while all other terms are elements of the corresponding linear space. In particular,

$$\Lambda(t) \approx \Lambda_0(\tau) + \sum_{\substack{i=-l\\ i\neq 0}}^{\infty} (t-\tau)^i Q_i(\tau), \tag{1}$$

where $\Lambda_0(\tau) \in \Lambda(\tau)^{\pitchfork}$, $Q_i(\tau) \in Sym^2\Lambda(\tau)$.

Assume that the curve $\Lambda(\cdot)$ is ample. Then $\Lambda_0(\tau) \in \Lambda(\tau)^{\pitchfork}$ is defined for all τ. The curve $\tau \mapsto \Lambda_0(\tau)$ is called the *derivative curve* of $\Lambda(\cdot)$.

Another characterization of $\Lambda_0(\tau)$ can be done in terms of the curves $t \mapsto \langle \Delta, \Lambda(t), \Lambda(\tau)\rangle$ in the linear space $Hom(\Delta, \Lambda(\tau))$, $\Delta \in \Lambda(\tau)^{\pitchfork}$. These curves have poles at τ. The Laurent expansion at $t = \tau$ of the vector function $t \mapsto \langle \Delta, \Lambda(t), \Lambda(\tau)\rangle$ has zero free term if and only if $\Delta = \Lambda_0(\tau)$.

The coordinate version of the series (2.1) is the Laurent expansion of the matrix-valued function $t \mapsto (S_t - S_\tau)^{-1}$ at $t = \tau$, where $\Lambda(t) = \{(x, S_t x) : x \in \mathbb{R}^n\}$.

3 Curvature operator and regular curves.

Using derivative curve one can construct an operator invariant of the curve $\Lambda(t)$ at any its point. Namely, take velocities $\dot\Lambda(t)$ and $\dot\Lambda_0(t)$ of $\Lambda(t)$ and its derivative curve $\Lambda_0(t)$. Note that $\dot\Lambda(t)$ is linear operator from $\Lambda(t)$ to $\Lambda(t)^*$ and $\dot\Lambda_0(t)$ is linear operator from $\Lambda_0(t)$ to $\Lambda_0(t)^*$. Since the form σ defines the canonical isomorphism between $\Lambda_0(t)$ and $\Lambda(t)^*$, the following operator $R(t) : \Lambda(t) \to \Lambda(t)$ can be defined:

$$R(t) = -\dot\Lambda_0(t) \circ \dot\Lambda(t) \tag{2}$$

This operator is called *curvature operator* of Λ at t. Note that in the case of Riemannian geometry the operator $R(t)$ is similar to the so-called Ricci operator $v \to R^\nabla(\dot\gamma(t), v)\dot\gamma(t)$, which appears in the classical Jacobi equation $\nabla_{\dot\gamma(t)}\nabla_{\dot\gamma(t)}V + R^\nabla(\dot\gamma(t), V)\dot\gamma(t) = 0$ for Jacobi vector fields V along the geodesic $\gamma(t)$ (here R^∇ is curvature tensor of Levi-Civita connection ∇), see [1]. This is the reason for the sign "$-$" in (2).

The curvature operator can be effectively used in the case of so-called regular curves. The curve $\Lambda(t)$ in Lagrange Grassmannian is called *regular*, if the quadratic form $\dot\Lambda(t)$ is nondegenerated for all t. Suppose that the curve $\Lambda(\cdot)$ is regular and has a coordinate representation $\Lambda(t) = \{(x, S_t x) : x \in \mathbb{R}^n\}$, $S_\tau = 0$. Then the function $t \mapsto S_t^{-1}$ has a simple pole at $t = \tau$ and one can get the following formula for the curvature operator (see [1]):

$$R(t) = \left((2S_t')^{-1}S_t''\right)' - \left((2S_t')^{-1}S_t''\right)^2 \tag{3}$$

Note that the right-hand side of (3) is a matrix analog of so-called Schwarz derivative or Schwarzian . Let us recall that the differential operator:

$$\mathbb{S} : \varphi \mapsto \frac{1}{3}\Big(\frac{d}{dt}\big(\frac{\varphi''}{2\,\varphi'}\big) - \Big(\frac{\varphi''}{2\,\varphi'}\Big)^2\Big) = \frac{1}{6}\frac{\varphi^{(3)}}{\varphi'} - \frac{1}{4}\Big(\frac{\varphi''}{\varphi'}\Big)^2, \tag{4}$$

acting on scalar function φ is called *Schwarzian.* The operator $\mathbb{S}$ is characterized by the following remarkable property: General solution of the equation $\mathbb{S}\varphi = \rho$ w.r.t φ is a Möbius transformation (with constant coefficients) of some particular solution of this equation. The matrix analog of this operator has similar property, concerning "matrix Möbius transformation" of the type $(AS+B)(CS+D)^{-1}$. It implies that in the regular case the curvature operator $R(t)$ determines the curve completely up to a symplectic transformation.

4 Expansion of the cross-ratio and Ricci curvature.

For the nonregular curve $\Lambda(t)=\{(x,S_tx): x\in\mathbb{R}^n\}$, the function $t\mapsto (S_t-S_\tau)^{-1}$ has a pole of order greater than 1 at τ and it is much more difficult to compute its Laurent expansion. In particular, as we will see later in the nonregular case the curvature operator does not determine the curve up to a symplectic transformation. However, using the notion of cross-ratio it is possible to construct numerical invariants for a very nonrestrictive class of curves.

Suppose that Λ_0, Λ_1, Λ_2, and Λ_3 are Lagrangian subspaces of W and $\Lambda_0\cap\Lambda_2=\Lambda_1\cap\Lambda_2=\Lambda_3\cap\Lambda_0=0$. We have $\langle\Lambda_0,\Lambda_1,\Lambda_2\rangle:\Lambda_0\to\Lambda_2$, $\langle\Lambda_2,\Lambda_3,\Lambda_0\rangle:\Lambda_2\to\Lambda_0$. The cross-ratio $\Big[\Lambda_0,\Lambda_1,\Lambda_2,\Lambda_3\Big]$ of four "points" Λ_0, Λ_1, Λ_2, and Λ_3 in the Lagrange Grassmannian is, by definition, the following linear operator in Λ_2:

$$\Big[\Lambda_0,\Lambda_1,\Lambda_2,\Lambda_3\Big]=\langle\Lambda_0,\Lambda_1,\Lambda_2\rangle\langle\Lambda_2,\Lambda_3,\Lambda_0\rangle. \tag{5}$$

This notion is a "matrix" analog of the classical cross-ratio of four points in the projective line. Indeed, let $\Lambda_i=\{(x,S_ix): x\in\mathbb{R}^n\}$, then, in coordinates $\{x\}$, the cross-ratio takes the form:

$$\Big[\Lambda_0,\Lambda_1,\Lambda_2,\Lambda_3\Big]=(S_2-S_1)^{-1}(S_1-S_0)(S_0-S_3)^{-1}(S_3-S_2) \tag{6}$$

By construction, all coefficients of the characteristic polynomial of $\Big[\Lambda_0,\Lambda_1,\Lambda_2,\Lambda_3\Big]$ are invariants of four subspaces $\Lambda_0,\Lambda_1,\Lambda_2$, and Λ_3.

Now we are going to show how to use the cross-ratio in order to construct invariants of the curve $\Lambda(t)$ in the Lagrange Grassmannian. Let, as before, $t\mapsto\{(x,S_tx): x\in\mathbb{R}^n\}$ be the coordinate representation of a germ of the curve $\Lambda(\cdot)$.

Assumption 1 *For all parameters t_1 the functions $t\to det(S_t-S_{t_1})$ have at $t=t_1$ zero of the same finite order k.*

By the above the function $(t_0,t_1,t_2,t_3)\to\det\Big[\Lambda(t_0),\Lambda(t_1),\Lambda(t_2),\Lambda(t_3)\Big]$ is symplectic invariant of the curve $\Lambda(t)$. Using this fact, let us try to find symplectic invariants of $\Lambda(t)$ that are functions of t. For this it is very convenient to introduce the following function

$$\mathcal{G}(t_0,t_1,t_2,t_3)=\ln\left|\frac{det\Big[\Lambda(t_0),\Lambda(t_1),\Lambda(t_2),\Lambda(t_3)\Big]}{\Big[t_0,t_1,t_2,t_3\Big]^k}\right|, \tag{7}$$

where $\left[t_0, t_1, t_2, t_3\right] = \frac{(t_0-t_1)(t_2-t_3)}{(t_1-t_2)(t_3-t_0)}$ is the usual cross-ratio of four numbers t_0, t_1, t_2,and t_3. The function $\mathcal{G}(t_0, t_1, t_2, t_3)$ is also a symplectic invariant of $\Lambda(t)$. It can be easily expanded in formal Taylor series at any "diagonal" point (t, t, t, t) and the coefficients of this expansion are invariants of the germ of $\Lambda(\cdot)$ at t.

Indeed, by Assumption 1, we have:

$$det(S_{t_0} - S_{t_1}) = (t_0 - t_1)^k X(t_0, t_1), \quad X(t,t) \neq 0 \tag{8}$$

for any t. It follows that $X(t_0, t_1)$ is a symmetric function (changing the order in (8) we obtain that X can be symmetric or antisymmetric, but the last case is impossible by the fact that $X(t,t) \neq 0$). Define also the following function

$$f(t_0, t_1) = ln|X(t_0, t_1)| \tag{9}$$

This function is also symmetric, so it can be expanded in the formal Taylor series at the point (t, t) in the following way:

$$f(t_0, t_1) \approx \sum_{i,j=0}^{\infty} \alpha_{i,j}(t)(t_0 - t)^i (t_1 - t)^j, \qquad \alpha_{i,j}(t) = \alpha_{j,i}(t) \tag{10}$$

One can easily obtain the following lemma, using the fact that

$$\mathcal{G}(t_0, t_1, t_2, t_3) = f(t_0, t_1) - f(t_1, t_2) + f(t_2, t_3) - f(t_3, t_0)$$

Lemma 1. *For any t the function $\mathcal{G}(t_0, t_1, t_2, t_3)$ has the following formal Taylor expansion at the point (t, t, t, t):*

$$\mathcal{G}(t_0, t_1, t_2, t_3) \approx \sum_{i,j=1}^{\infty} \alpha_{i,j}(t)(\xi_0^i - \xi_2^i)(\xi_1^j - \xi_3^j), \tag{11}$$

where $\xi_l = t_l - t$, $l = 0, 1, 2, 3$.

From (11) it follows that all coefficients $\alpha_{i,j}(t)$, $i, j \geq 1$, are symplectic invariants of the curve $\Lambda(t)$.

Definition 1. The first appearing coefficient $\alpha_{1,1}(t)$ is called Ricci curvature of $\Lambda(t)$.

In the sequel the Ricci curvature is denoted by $\rho(t)$. Note that, by a direct computation, one can get the following relation between $\rho(\tau)$ and curvature operator $R(\tau)$ for the regular curve: $\rho(\tau) = \frac{1}{3} tr\, R(\tau)$. Actually , this relation justifies the name Ricci curvature for the invariant $\rho(t)$.

In some cases we are interested in symplectic invariants of unparametrized curves in Lagrange Grassmannian (i.e., of one-dimensional submanifolds in Lagrange Grassmannian). For example, so-called abnormal extremals of vector distributions and consequently their Jacobi curves a priori have no special parametrizations.

Now we want to show how, using the Ricci curvature, one can define a canonical projective structure on the unparametrized curve $\Lambda(\cdot)$. For this let us check how the Ricci curvature is transformed by a reparametrization of the curve $\Lambda(t)$.

Let $t=\varphi(\tau)$ be a reparametrization and let $\bar\Lambda(\tau)=\Lambda(\varphi(\tau))$. Denote by $\overline{\mathcal{G}}$ the function playing for $\bar\Lambda(\tau)$ the same role as the function $\mathcal{G}$ for $\Lambda(t)$. Then from (7) it follows that

$$\overline{\mathcal{G}}(\tau_0,\tau_1,\tau_2,\tau_3)=\mathcal{G}(t_0,t_1,t_2,t_3)+k\,\ln\left(\frac{[\varphi(\tau_0),\varphi(\tau_1),\varphi(\tau_2),\varphi(\tau_3)]}{[\tau_0,\tau_1,\tau_2,\tau_3]}\right),\qquad(12)$$

where $t_i=\varphi(\tau_i)$, $i=0,1,2,3$. By direct computation it can be shown that the function $(\tau_0,\tau_1,\tau_2,\tau_3)\mapsto\ln\left(\frac{[\varphi(\tau_0),\varphi(\tau_1),\varphi(\tau_2),\varphi(\tau_3)]}{[\tau_0,\tau_1,\tau_2,\tau_3]}\right)$ has the following Taylor expansion up to the order two at the point (τ,τ,τ,τ):

$$\ln\left(\frac{[\varphi(\tau_0),\varphi(\tau_1),\varphi(\tau_2),\varphi(\tau_3)]}{[\tau_0,\tau_1,\tau_2,\tau_3]}\right)=\mathbb{S}\varphi(\tau)(\eta_0-\eta_2)(\eta_1-\eta_3)+\ldots,\qquad(13)$$

where $\mathbb{S}\varphi$ is Schwarzian defined by (4) and $\eta_i=\tau_i-\tau$, $i=0,1,2,3$.

Suppose for simplicity that in the original parameter t the Ricci curvature $\rho(t)\equiv0$ and denote by $\bar\rho(\tau)$ the Ricci curvature of the curve $\bar\Lambda(\tau)$. Then from (11), (12), and (13) it follows easily that:

$$\bar\rho(\tau)=k\,\mathbb{S}\varphi(\tau).\qquad(14)$$

Conversely, if the Ricci curvature $\rho(t)$ of the curve $\Lambda(t)$ is not identically zero we can find at least locally (i.e., in a neighbourhood of given point) a reparametrization $t=\varphi(\tau)$ such that $\bar\rho(\tau)\equiv0$ (from (14) it follows that in this case $\varphi(\tau)$ has to satisfy the equation $\mathbb{S}(\varphi^{-1})(t)=\frac{\rho(t)}{k}$)

The set of all parametrization of $\Lambda(\cdot)$ with Ricci curvature identically equal to zero defines a projective structure on $\Lambda(\cdot)$ (any two parametrization from this set are transformed one to another by Möbius transformation). We call it *the canonical projective structure* of the curve $\Lambda(\cdot)$. The parameters of the canonical projective structure will be called *projective parameters*.

5 Fundamental form of the unparametrized curve.

The Ricci curvature $\rho(\cdot)$ is the first coefficient in the Taylor expansion of the function $\mathcal{G}$ at the point (t,t,t,t). The analysis of the next terms of this expansion gives the way to find other invariants of the curve $\Lambda(\cdot)$ that do not depend on $\rho(t)$. In this section we show how to find candidates for the "second" invariant of $\Lambda(\cdot)$ and then we construct a special form on unparametrized curve $\Lambda(\cdot)$ (namely, the differential of order four on $\Lambda(\cdot)$), which we call the fundamental form of the curve $\Lambda(\cdot)$.

First note that analyzing the expansion (10) one can easily obtain the following lemma

Lemma 2. *Let $\alpha_{i,j}(t)$ be as in expansion (10). Then the following relation holds*

$$\alpha'_{i,j}(t) = (i+1)\alpha_{i+1,j}(t) + (j+1)\alpha_{i,j+1}(t) \tag{15}$$

In particular, from (15) it follows easily that

$$\alpha_{2,1}(t) = \frac{1}{4}\rho'(t), \quad \alpha_{2,2}(t) = \frac{1}{8}\rho''(t) - \frac{3}{2}\alpha_{3,1}(t) \tag{16}$$

These relations imply that the function $\alpha_{3,1}(t)$ is a candidate for the second invariant (as well as the function $\alpha_{2,2}(t)$).

Now let t be a projective parameter on $\Lambda(\cdot)$. Then by definition $\rho(t) \equiv 0$, and by (16) $\alpha_{2,1}(t) \equiv 0$ and $\alpha_{2,2}(t) = -\frac{3}{2}\alpha_{3,1}(t)$. This together with (11) and the fact that $\alpha_{3,1}(t) = \alpha_{1,3}(t)$ implies that the function $\mathcal{G}(t_0, t_1, t_2, t_3)$ has the following Taylor expansion up to the order four at the point (t, t, t, t):

$$\mathcal{G}(t_0, t_1, t_2, t_3) = \alpha_{3,1}(t) p_4(\xi_0, \xi_1, \xi_2, \xi_3) + \dots, \tag{17}$$

where $\xi_i = t_i - t$, $i = 0, 1, 2, 3$, and $p_4(\xi_0, \xi_1, \xi_2, \xi_3)$ is a homogeneous polynomial of degree four (more precisely, $p_4(\xi_0, \xi_1, \xi_2, \xi_3) = (\xi_0^3 - \xi_2^3)(\xi_1 - \xi_3) + (\xi_0 - \xi_2)(\xi_1^3 - \xi_3^3) - \frac{3}{2}(\xi_0^2 - \xi_2^2)(\xi_1^2 - \xi_3^2))$.

Let τ be another projective parameter on $\Lambda(\cdot)$ (i.e., $t = \varphi(\tau) = \frac{a\tau + b}{c\tau + d}$) and denote by $\bar{\alpha}_{3,1}(\tau)$ the function that plays the same role for the curve $\Lambda(\varphi(\tau))$ as the function $\alpha_{3,1}(t)$ for $\Lambda(t)$. Then from (12), (17), and the fact that the cross-ratio is preserved by Möbius transformations it follows that

$$\bar{\alpha}_{3,1}(\tau)(d\tau)^4 = \alpha_{3,1}(t)(dt)^4 \tag{18}$$

It means that the form $\alpha_{3,1}(t)(dt)^4$ does not depend on the choice of the projective parameter t. We will call this form *a fundamental form* of the curve $\Lambda(\cdot)$.

If t is an arbitrary (not necessarily projective) parameter on the curve $\Lambda(\cdot)$, then the fundamental form in this parameter has to be of the form $A(t)(dt)^4$, where $A(t)$ is a smooth function (the "density" of the fundamental form). For projective parameter $A(t) = \alpha_{3,1}(t)$. For arbitrary parameter it can be shown, using (11), (12), that $A(t) = \alpha_{3,1}(t) - \frac{1}{5k}\rho(t)^2 - \frac{1}{20}\rho''(t)$.

If $A(t)$ does not change sign, then the canonical length element $|A(t)|^{\frac{1}{4}}dt$ is defined on $\Lambda(\cdot)$. The corresponding parameter τ (i.e., length with respect to this length element) is called *a normal parameter* (in particular, it implies that abnormal extremals of vector distribution may have canonical (normal) parametrization). Calculating the Ricci curvature $\rho_n(\tau)$ of $\Lambda(\cdot)$ in the normal parameter, we obtain a functional invariant of the unparametrized curve. We will call it *projective curvature* of the unparametrized curve $\Lambda(\cdot)$. If $t = \varphi(\tau)$ is the transition function between a projective parameter t and the normal parameter τ, then by (14) it follows that $\rho_n(\tau) = k\,\mathbb{S}\varphi(\tau)$.

Note that all constructions of this section can be done for the curve in the Grassmannian $G(m, 2m)$ (the set of all m-dimensional subspaces in the $2m$-dimensional linear space) instead of Lagrange Grassmannian by the action of the group $GL(2m)$ instead of Symplectic Group.

6 The method of moving frame.

In this section we consider nonregular curves having two functional invariants and prove that the above defined invariants ρ and A constitute a complete system of symplectic invariants for these curves (completeness of the system of invariants means that the system defines the curve uniquely up to a symplectic transformation)

Assume that dimension of the symplectic space W is four and consider ample curves $\Lambda(t)$ in $L(W)$ such that for any t the velocity $\dot\Lambda(t)$ is a quadratic form of rank 1. Without loss of generality we can assume that $\dot\Lambda(t)$ is nonnegative definite for any t. Let us fix some parameter τ and let $\{(x, S_t x) : x \in \mathbb{R}^n\}$ be a coordinate representation of $\Lambda(t)$ such that $S_\tau = 0$. Since the curve $\Lambda(t)$ is ample, the curve S_t^{-1} has a pole at τ. The velocity $\frac{d}{dt}S_t^{-1} : \Lambda(\tau)^* \to \Lambda(\tau)$ is a well defined self-adjoint operator. Moreover, by our assumptions, $\frac{d}{dt}S_t^{-1}$ is a nonpositive self-adjoint operator of rank 1. So for $t \neq \tau$ there exists unique, up to the sign, vector $w(t,\tau) \in \Lambda(\tau)$ such that for any $v \in \Lambda(\tau)^*$

$$\langle v, \frac{d}{dt}S_t^{-1} v\rangle = -\langle v, w(t,\tau)\rangle^2 \tag{19}$$

It is clear that the curve $t \to w(t,\tau)$ also has the pole at τ. Suppose that the order of the pole is equal to l. Denote by $u(t,\tau)$ the normalized curve $t \to u(t,\tau) = (t-\tau)^l w(t,\tau)$ and define the following vectors in $\Lambda(\tau)$: $e_j(\tau) = \frac{\partial^{j-1}}{\partial t^{j-1}} u(t,\tau)\Big|_{t=\tau}$.

It is not hard to show that the order of pole of $t \to w(t,\tau)$ at $t = \tau$ is equal to l if and only if l is the minimal positive integer such that the vectors $e_1(\tau)$ and $e_l(\tau)$ are linear independent (in particular, $e_1(\tau)$ and $e_2(\tau)$ are linear independent if and only if $l = 2$). It implies easily that the set of points τ, where the vectors $e_1(\tau)$ and $e_2(\tau)$ are linear dependent, is a set of isolated points in $\mathbb{R}$.

Assumption 2 *$\Lambda(t)$ is a curve in $L(W)$ with $\dim W = 4$ such that for any t the velocity $\dot\Lambda(t)$ is a quadratic form of rank 1 and $e_1(t)$, $e_2(t)$ are linear independent.*

By the above it is easy to see that if $\Lambda(\cdot)$ satisfies the previous assumption, then it satisfies Assumption 1 with $k = 4$. So, the invariants $\rho(\cdot)$ and $A(\cdot)$ are defined for $\Lambda(\cdot)$. Note that the curve $\Lambda(\cdot)$ can be described by the curve $t \to w(t,\tau)$ of the vectors on the plane, i.e. $\Lambda(\cdot)$ can be described by two functions. The natural question is whether $(\rho(\cdot), A(\cdot))$ is a complete system of symplectic invariants of $\Lambda(\cdot)$.

Since vector $w(t,\tau)$ is defined up to the sign, the vector $e_1(\tau)$ is also defined up to the sign. So, for any τ one can take $(e_1(\tau), e_2(\tau))$ or $(-e_1(\tau), -e_2(\tau))$ as the canonical bases on the plane $\Lambda(\tau)$. Recall that by constructions of the section 2 for the curve $\Lambda(\cdot)$ the derivative curve $\Lambda_0(\cdot)$ is defined and for any τ the subspaces $\Lambda(\tau)$ and $\Lambda_0(\tau)$ are transversal. So, in addition to the vectors

$e_1(\tau), e_2(\tau)$ on the plane $\Lambda(\tau)$, one can choose two vectors $f_1(\tau)$ and $f_2(\tau)$ on the plane $\Lambda_0(\tau)$ such that four vectors $\Big(e_1(\tau), e_2(t), f_1(\tau), f_2(\tau)\Big)$ constitute symplectic basis (or Darboux basis) of W (it means that $\sigma(f_i(\tau), e_j(\tau)) = \delta_{i,j}$). So, the curve $\Lambda(\cdot)$ defines a moving frame $\Big(e_1(\tau), e_2(\tau), f_1(\tau), f_2(\tau)\Big)$ and one can derive the structural equation for this frame:

Proposition 2. *The frame* $\Big(e_1(\tau), e_2(\tau), f_1(\tau), f_2(\tau)\Big)$ *satisfies the following structural equation:*

$$\begin{cases} \dot e_1 = 3e_2 \\ \dot e_2 = \frac{1}{4}\rho e_1 + 4f_2 \\ \dot f_1 = -(\frac{35}{12}A - \frac{1}{8}\rho^2 + \frac{1}{16}\rho'')e_1 - \frac{7}{16}\rho' e_2 - \frac{1}{4}\rho f_2 \\ \dot f_2 = -\frac{7}{16}\rho' e_1 - \frac{9}{4}\rho e_2 - 3f_1 \end{cases} \tag{20}$$

Note that the coefficients in the equation (20) depend only on ρ and A and any symplectic basis can be taken as an initial condition of (20). It implies the following:

Theorem 1. *The curve* $\Lambda(\cdot)$ *satisfying Assumption 2 is determined by its invariants* $(\rho(\cdot), A(\cdot))$ *uniquely up to the symplectic transformation of* W.

Remark 1. It can be shown by a direct calculation that the curvature operator $R(\tau) : \Lambda(\tau) \to \Lambda(\tau)$ of the curve $\Lambda(\cdot)$ satisfying Assumption 2 has the following matrix in the basis $(e_1(\tau), e_2(\tau))$: $R(\tau) = \begin{pmatrix} 0 & \frac{7}{4}\rho'(\tau) \\ 0 & 9\rho(\tau) \end{pmatrix}$, i.e., R depends only on ρ. This means that in contrast to the regular case, the curvature operator does not determine the curve $\Lambda(\cdot)$ uniquely up to a symplectic transformation.

Theorem 1 implies the following result on unparametrized curves:

Theorem 2. *Assume that the curve* $\Lambda(\cdot)$ *satisfies Assumption 2 for some parametrization and its fundamental form* $A(t)(dt)^4$ *does not vanish. Then the sign of* $A(t)$ *and the projective curvature* $\rho_n(\cdot)$ *determine* $\Lambda(\cdot)$ *uniquely up to a symplectic transformation of* W *and a reparametrization.*

7 Flat curves.

The following definition is natural.

Definition 2. The curve $\Lambda(t)$, satisfying Assumption 2, is called flat if $\rho(t) \equiv 0$, $A(t) \equiv 0$.

As a consequence of Theorem 1, expansion (11), and structural equation (20) one can obtain the following characterization of the flat curve:

Theorem 3. *The curve $\Lambda(t)$, satisfying Assumption 2, is flat if and only if one of the following condition holds:*

1) all coefficients $Q_i(t)$ with $i > 0$ in the Laurent expansion (1) are equal to zero;

2) the derivative curve $\Lambda_0(t)$ is constant, i.e., $\dot{\Lambda}_0(t) \equiv 0$;

3) for any t_0, t_1, t_2, t_3

$$det\Big([\Lambda(t_0), \Lambda(t_1), \Lambda(t_2), \Lambda(t_3)]\Big) = [t_0, t_1, t_2, t_3]^4. \qquad (21)$$

The conditions 1), 2), and 3) are also equivalent for regular curves in $L(W)$ with symplectic space W of arbitrary dimension (we only need to replace the power 4 in (21) by $\dim W$). In the last case these conditions are also equivalent to the fact that curvature operator $R(t)$ is identically equal to zero.

Conjecture. *Suppose that $\Lambda(t)$ satisfies Assumption 1. Then the conditions 1), 2), and 3), with the power 4 replaced by k in relation (21), are equivalent.*

If the previous conjecture is true, then one of the conditions 1), 2), or 3) can be taken as a definition of the flat curve.

Now let us discuss the notion of flatness for unparametrized curves.

Definition 3. An unparametrized curve $\Lambda(\cdot)$, satisfying Assumption 2 for some parametrization, is called flat, if its fundamental form is identically zero.

It happens that, up to symplectic transformations and reparametrizations, there exists exactly one maximal flat curve.

Theorem 4. *There is an embedding of the real projective line $\mathbb{RP}^1$ into $L(W)$ as a flat closed curve endowed with the canonical projective structure; Maslov index of this curve equals 2. All other flat curves are images under symplectic transformations of $L(W)$ of the segments of this unique one.*

References

1. A.A. Agrachev, R.V. Gamkrelidze, Feedback-invariant optimal control theory - I. Regular extremals, J. Dynamical and Control Systems, **3**,1997, No. 3, 343-389.
2. A.A. Agrachev, Feedback-invariant optimal control theory - II. Jacobi Curves for Singular Extremals, J. Dynamical and Control Systems, **4**(1998), No. 4 , 583-604.
3. I. Zelenko, Nonregular abnormal extremals of 2-distribution: existence, second variation and rigidity, J. Dynamical and Control Systems, **5**(1999), No. 3, 347-383.